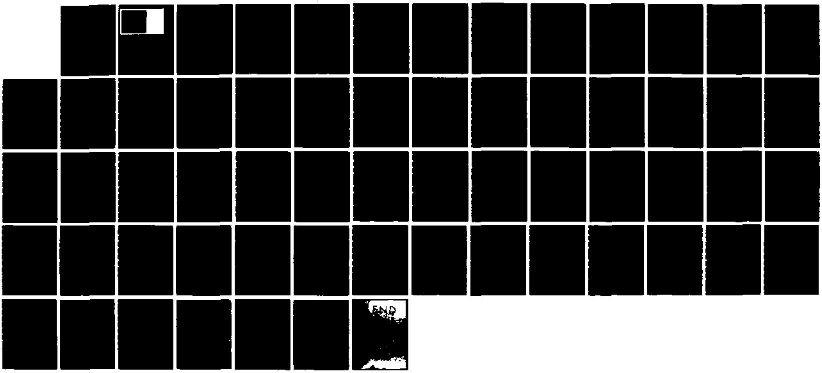
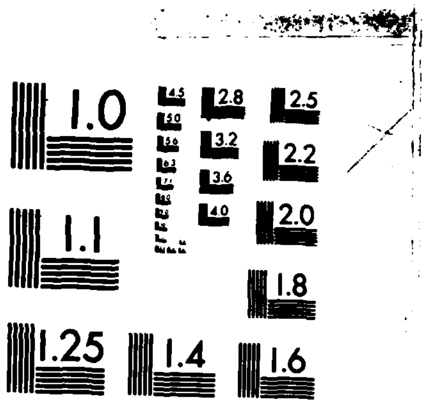
AD-A147 242 ASYMPTOTIC BEHAVIOUR OF THE PLASMA EQUATION(U) 1/1
WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER
Y C KWONG AUG 84 MRC-TSR-2727 DAAG29-80-C-0041

UNCLASSIFIED F/G 12/1 NL




1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0

3

MRC Technical Summary Report #2727

ASYMPTOTIC BEHAVIOUR OF THE PLASMA EQUATION

Y. C. Kwong

AD-A147 242

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

August 1984

(Received February 20, 1984)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

-a-

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

ASYMPTOTIC BEHAVIOUR OF THE PLASMA EQUATION

Y. C. Kwong

Technical Summary Report #2727
August 1984

ABSTRACT

In this paper, is we are concerned with a ~~the~~ plasma equation.

$v(x,t)_t = \Delta v(x,t)^{\frac{1}{q-1}}$ where $q > 2$, $t > 0$, $x \in \Omega$, Ω being a bounded smooth domain in R^N, with non-negative initial data and a homogeneous Dirichlet boundary condition. It is known that there exists a finite extinction time T^* such that the solution decays to zero at T^*. Recently, Berryman and Holland investigated the stability of the profile of the solution as approaches limit of $t \to T^*$. However, they obtained their results at the expense of some very strong regularity assumptions. In this paper, we prove s the same kind of results without those strong regularity assumptions. By invoking both the nonlinear "semi-group" theory and a standard regularizing scheme for the equation, the author ~~we~~ measures the rate of decay of the solution and obtains estimates on the time derivative as $t \to T^*$. As motivated by the regularity assumptions, both the interior and boundary regularities of the solution are studied. Finally, he ~~we~~ perturbs the nonlinearity of the plasma equation and studies ~~study~~ the same aspects for the perturbed equation.

AMS (MOS) Subject Classification: 35K15, 35K20, 35K55, 35K65

Key Words: nonlinear degenerate diffusion equation, plasma equation, extinction time, stability, nonlinear semi-group, interior and boundary regularity

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

It is well known that for a certain class of nonlinear degenerate parabolic equations, of which the plasma equation is an important example, there exists a finite extinction time T^* for certain boundary conditions. It is very natural and interesting to investigate the shape of the solutions of such equations near extinction. However, very little work has been done on this problem, probably due to the fact that it is difficult to obtain sharp estimates for the decay rate as $t \to T^*$. Recently, Berryman and Holland have obtained results concerning the stability of the profile of the separable solution of the equation. However, their work is based on some very strong regularity assumptions on the solution (see [12] for the details). It is therefore essential to weaken these assumptions as much as possible. Indeed, in this paper we prove the same results without involving these regularity assumptions. We also investigate the interior and boundary regularity of the solution; we show that the solution is a positive classical solution on $\Omega \times (0,T^*)$ and is continuous up to the boundary. DiBenedetto in [7] mentioned that the continuity up to the boundary for the plasma equation can be proved in a similar way as in his proof of the porous medium equation. The virtue of the proof in this paper is its simplicity and its *explicitness*. (We obtain an explicit estimate of the "Lipschitz rate" of decay of the solution at the boundary). Of course, such regularity results are not only true for the plasma equation, but they also hold for the solutions of equations having certain types of nonlinearities.

Furthermore, since our work uses both the nonlinear "semi-group" theory and the traditional notion of weak solution, we have to establish that the notion of weak solution coincides with the unique semi-group solution, and we do so by a theorem concerning the uniqueness of the weak solution.

Finally, we show that the same results hold even when we perturb the nonlinearity of the plasma equation by a sufficiently small amount. (This will be made precise in Section IV.)

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

Notations:

Ω : A bounded domain with smooth boundary in R^N.

$\partial\Omega$: The boundary of Ω.

Q_T : The parabolic cylinder $\Omega \times (0,T)$.

η : The outward normal at the boundary $\partial\Omega$.

$C_0^\infty(\Omega)$: The space of all infinitely differentiable functions with compact support in Ω.

$W_p^\ell(\Omega)$ for ℓ integral: The Banach space consisting of all elements of $L^p(\Omega)$ whose generalized derivatives of order up to and including ℓ are in $L^p(\Omega)$ with the norm,

$$\|u\|_{W_p^\ell(\Omega)} = \sum_{k=1}^{\ell} \|D_x^k u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}$$

$D_x^k u$ stands for the generalized derivatives of u of order k.

$\overset{\circ}{W}{}_p^\ell(\Omega)$: The closure of $C_0^\infty(\Omega)$ in the norm $\| \ \|_{W_p^\ell(\Gamma)}$.

$W^{1,1}(Q_T)$: $\{u \in L^2(Q_T) | \nabla u, u_t \in L^2(Q_T)\}$ with the norm

$$\|u\|_{W^{1,1}(Q_T)} = \sqrt{\|u\|^2_{L^2(Q_T)} + \|u_t\|^2_{L^2(Q_T)} + \|\nabla u\|^2_{L^2(Q_T)}}$$

$\overset{\circ}{W}{}^{1,1}(Q_T)$: The closure of smooth functions on Q_T vanishing on $\partial\Omega \times [0,T]$ in the norm $\| \ \|_{W^{1,1}(Q_T)}$.

$\overset{\circ}{V}_2(Q_T)$: $L^\infty(0,T|L^2(\Omega)) \cap L^2(0,T|\overset{\circ}{W}{}_2^1(\Omega))$ with the norm

$$\|u\|_{V_2(Q_T)} = \sqrt{\sup_{0\le t\le T} \|u(\cdot,t)\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(Q_T)}} \quad \text{for } u \in \overset{\circ}{V}_2(Q_T) .$$

$L^r(0,T|L^s(\Omega))$: The Banach space consisting of all measurable functions on Q_T with

$$\|u\|_{L^r(0,T|L^s(\Omega))} = \left[\int_0^T \left(\int_\Omega |u(x,t)|^s dx\right)^{r/s} dt\right]^{1/r} < \infty .$$

$C^{2,1}(Q_T)$: The Banach space of all continuous functions in Q_T having derivatives u_x, u_{xx}, u_t in $C(Q_T)$ with $\| \ \|_{C^{2,1}(Q_T)}$ defined as usual.

$H^{\ell,\ell/2}(\bar{Q}_T)$: The Banach space of functions $u(x,t)$ that are continuous in $\bar{Q}_T$, together with all derivatives of the form $D_t^r D_x^s u$ for $2r + s < \ell$ with r and s integral. For $2r + s = [\ell]$, $D_t^r D_x^s u$ are Hölder continuous in x with Hölder exponent $\ell - [\ell]$ and for $0 < \ell - 2r - s < 2$, $D_t^r D_x^s u$ are Hölder continuous in t with Hölder exponent $\frac{\ell - 2r - s}{2}$. The norm $\| \ \|_{H^{\ell,\ell/2}(\bar{Q}_T)}$ is defined as in [26].

$H^{\ell,\ell/2}(Q_T)$: The set of functions belonging to $H^{\ell,\ell/2}(\bar{Q}_T')$ where Q_T' is any proper subset of Q_T. Finally, let X be any Banach space with its norm denoted by $\| \ \|_X$.

$C([0,T]|X)$: The Banach space of all continuous X-valued functions $u(t)$ with domain $[0,T]$ and $\|u\|_{C([0,T]|X)} = \sup_{0 \le t \le T} \|u(t)\|_X$.

$L^r(0,T|X)$: The class of X-valued functions $u(t)$ which are strongly measurable and $t \to \|u(t)\| \in L^r(0,T)$ with $\|u\|_{L^r(0,T|X)} = \left(\int_0^T \|u(t)\|_X^r \, dt\right)^{1/r}$.

$AC_{loc}(0,T)$: The set of all those functions which are locally absolutely continuous on $(0,T)$.

"$\to$" and "$\rightharpoonup$" denote respectively strong and weak convergence in Banach spaces.

Remark: The subscript loc on the spaces $L^r(0,T|L^s(\Omega))$ refers to local integrability.

ASYMPTOTIC BEHAVIOUR OF THE PLASMA EQUATION

Y. C. Kwong

INTRODUCTION

Let Ω be a smooth domain in $\mathbb{R}^N$, bounded or unbounded. The nonlinear diffusion equation

$$(1) \qquad \Delta v(x,t)^m = v_t(x,t), \quad x \in \Omega, \quad t > 0, \quad m > 0$$

has been studied by many authors. The case where $m > 1$ is known as the porous media equation. Regularity for the porous media equation has been widely studied by D. Aronson in [1,2,3], Caffarelli and Friedman in [18, 19] (here $\Omega = \mathbb{R}^N$ and an explicit modulus of continuity of the Hölder type has been found), Evans and Caffarelli in [17], DiBenedetto in [7] and Paul Sacks in [30, 31]. The asymptotic behaviour as $t \to \infty$ as has been studied by several authors including Aronson and Peletier in [5], Bertsch and Peletier in [14] J. L. Vazquez in [33]. (In the latter two cases, $\Omega = \mathbb{R}^N$).

In this paper however, we are interested in the case where Ω is a bounded smooth domain in $\mathbb{R}^N$ and $0 < m < 1$, when (1) is known as the fast diffusion equation. (It is also called the plasma equation because it arises in the modelling of ionized gases in plasma physics). Both the regularity and the asymptotic behaviour of the plasma equation are less well understood than for the porous media equation. Regularity has been studied by E. Sabanina in [29] for $N = 1$. In this paper, she proved the existence of positive classical solutions of (1) satisfying Dirichlet conditions and assuming non-negative initial data. For $N > 1$, Paul Sacks in [30] and DiBenedetto in [7] proved the existence of a bounded continuous weak solution. However no explicit estimate of the modulus of continuity has been found yet.

As far as the asymptotic behaviour is concerned, the main difference between the fast diffusion case and the porous media equation case (which is sometimes known as the slow diffusion equation) is that for the former case, there exists an extinction time T^*, i.e. a finite time T^* such that the solution dies down to zero at T^*. The existence of T^*

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

has been studied by D. Diaz in [22], E. Sabanina in [29], Crandall and Benilan in [8], Herrero and Vazquez in [25]. A natural problem would then be to investigate the asymptotic profile of the solution as $t \to T^*$. (The case where $\Omega = (0,\infty)$ with positive constant Dirichlet condition, the behaviour of the solution as $t \to \infty$ has been studied by Bertsch in [13]).

For the case where Ω is a bounded smooth domain in $\mathbb{R}^N$, Berryman and Holland [12], using strong regularity assumptions, have obtained some results concerning the stability of the profile of the separable solution of the equation. By separable solution, we are referring to a solution of the form $S(x)T(t)$ where x and t are the space and time variables respectively. In order that the function $S(x)T(t)$ is a positive separable solution of (2) in $\Omega \times (0,T^*)$ satisfying the zero lateral boundary condition and vanishing at $t = T^*$, we need $T(t) = [(q-2)(T^* - t)]^{\frac{1}{q-2}}$ and S to be a solution of;

(3) $$\Delta S = -(q-1)S^{q-1}, \quad S\big|_{\partial\Omega} = 0, \quad S > 0 \ \text{in} \ \Omega .$$

We note that if $q > 2$ for $N \leq 2$ or $2 < q < \frac{2N}{N-2}$ for $N > 2$. Then a positive classical solution of (3) does exists and lies in $C^{2+\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. (A summary of the uniqueness and existence theorems of *(3)* can be found in *[12]*).

We now give a listing of the technical hypothese and results in [12].

To facilitate computations, we let $m = 1/(q-1)$ where $q > 2$. We look at the nonlinear diffusion equation,

(2) $$\begin{cases} \Delta v^{\frac{1}{q-1}} = v_t, \quad v\big|_{\partial\Omega} = 0 \\ v(x,0) = v_0(x) \geq 0, \quad v_0 \not\equiv 0 \ \text{and} \ v_0\big|_{\partial\Omega} = 0 \end{cases}$$

alternatively,

(2') $$\begin{cases} \Delta u = (u^{q-1})_t, \quad u\big|_{\partial\Omega} = 0 \\ u(x,0) = u_0(x) = v_0(x)^{\frac{1}{q-1}} \end{cases}$$

here again, $2 < q < \infty$ and Ω is a smooth bounded domain in R^N. The hypothese of [12] in addition to the existence of T^* are as follows:

[H1] $u(x,t)$ is a classical solution of (2') on $\Omega \times (0,T^*)$ which is positive on $\Omega \times (0,T^*)$ and assumes the initial and boundary values.

[H2] There exists a constant γ with $T^* > \gamma > 0$, such that for any $T' > T^*$, u_t, u_{xi}, u_{xit} and u_{xixi} lie in $C(\overline{\Omega \times (\gamma,T')})$.

Using these assumptions, Berryman and Holland proved the following results.

<u>Main Theorem</u>: Let $p(x,t) = u(x,t)/((q - 2)(T^* - t))^{\frac{1}{q-2}}$ where $u(x,t)$ is the solution of (2'), then,

(i) If $2 < q$ for $N > 2$ or $2 < q < \frac{2N}{N - 2}$ for $N > 2$, there exists an increasing sequence of times $t_n \to T^*$ and a positive classical solution of (3) such that $p(\cdot,t_n) \to S(\cdot)$ in $\overset{\circ}{W}{}_2^1(\Omega)$ as $t_n \to T^*$.

(ii) If $q > \frac{2N}{N - 2}$, $N > 2$, then there is a non-negative weak solution $S(\cdot)$ of (3) and an increasing sequence of times $t_n \to T^*$ such that $p(\cdot,t_n) \to S(\cdot)$ in $\overset{\circ}{W}{}_2^1(\Omega)$.

The aim of this paper is to prove these results without these strong regularity assumptions. Also, motivated by these assumptions, the regularity of the solution $u(x,t)$ is investigated. Indeed, we will prove that $u(x,t)$ is a positive classical solution of (2') on $\Omega \times (0,T^*)$ if we make reasonable assumptions on the initial data u_0 and q (As mentioned earlier, this result has been proved by E. Sabanina in [29] for $n = 1$). Continuity up to the boundary is also studied under further assumptions on $\partial\Omega$, the boundary of Ω. (Continuity up to the boundary has also been studied by DiBenedetto in [7].)

In [12], the Main Theorem is a natural consequence of three key estimates, we will call these estimates Lemmas I, II and III in this paper.

To obtain these estimates, we make use of our knowledge from the semi-group theory concerning, regularity properties of the semi-group solution of (2), continuous dependence theorems of the semi-group solution, etc. To investigate the interior regularity of the

solution of (2), we use the notion of the bounded continuous weak solution and our knowledge about it.

To justify using the notion of the semi-group solution and the notion of weak solution at the same time, we use the uniqueness of weak solutions of (2) and (2') as a tool to show that in fact, these two notions of solutions are equivalent. (These two notions of solutions and their relationships will be made clear in the preliminaries).

Finally, we perturb the nonlinearity of the plasma equation and investigate the same asymptotic behaviour.

We now give an outline of the paper as follows:

Section (I): This consists of preliminaries which contain the necessary background material for this paper.

Section (II): Here, using Evan's results about the semi-group solution of (2), we prove the existence of the extinction time T^* and Lemma I. The existence of T^* follows as a natural consequence. (The existence of T^* has been proved by several authors as mentioned before). We remark that throughout (II), (III), we will restrict our initial data u_0 to be in $L^\infty(\Omega)$, nonnegative and not identically zero, we will relax this restriction starting in Section (IV).

Section (III): In this section, we look at the regularized version of (2'), that is :

$$(2',n,\varepsilon)\qquad \begin{cases} \left({u_n^\varepsilon}^{q-1}\right)_t - \Delta u_n^\varepsilon = 0, \quad u_n^\varepsilon\big|_{\partial\Omega} = \varepsilon, \quad \varepsilon > 0 \\ \\ u_n^\varepsilon(x,0) = u_0^n(x) + \varepsilon \end{cases}$$

where u_0^n converges to u_0 in some suitable way (see Section (I)). We will show that u_n^ε converges in $C((0,T]|L^p(\Omega))$, $1 < p < \infty$ to $u = v^{\frac{1}{q-1}}$, v being the semi-group solution of (2). (This will be done in Section (IV)). Using this nice convergence property, we will prove the other two key estimates, Lemma II and Lemma III, which will then ultimately lead to the desired stability results.

Section (IV): This section is devoted completely to the convergence of u_n^ε. By proving a uniqueness theorem of the weak solution with $u_0 \in L^\infty(\Omega)$. Thus, the weak solution which is the limit of u_n^ε as $n \to \infty$, $\varepsilon \to 0$, is the semi-group solution. Also, for the stability results, we will release the restriction of $L^\infty(\Omega)$ initial data to $L^{q-1}(\Omega)$ initial data under additional hypothesis on the range of q. The main machinery will be the regularizing effect of the solution of (2).

Section (V): The interior regularity of (2') in the parabolic cylinder $\Omega \times (0,T^*)$ is examined using the regularization $(2',n\ \varepsilon)$ of (2'). The continuity of $u(x,t)$ up to the boundary of Ω is also examined under certain assumptions on $\partial\Omega$, the boundary of Ω.

Section (VI): In this section, we perturb the non-linearity of the plasma equation. We consider the more general equation:

$$(4) \qquad \begin{cases} \beta'(u)u_t - \Delta u = 0 & u\big|_{\partial\Omega} = 0 \\ u(0) = u_0 \end{cases}$$

where $\beta(0) = 0$ and there exists $M > m > 0$, $ms^{q-2} < \beta'(s) < Ms^{q-2}$. We will show that if $\beta'(s) \to (q-1)s^{q-2}$ as $s \to 0$ rapidly enough, we can generalize the previous results for (4). The main machinery in this section is the modulus of continuity of the solution of (4).

SECTION I. PRELIMINARIES AND BACKGROUND MATERIALS

(I) Weak solution:

Definition: Suppose the initial data u_0 is in $L^\infty(\Omega)$, let $E = \{\psi \in \mathring{H}^{1,1}(Q_T) \mid \psi(x,T) = 0 \text{ a.e. } x \in \Omega\}$, an element u of $\mathring{V}_2(Q_T)$ is a weak solution of (2') if $u^{q-1} \in L^2(Q_T)$ and it satisfies:

$$\iint_{Q_T} u^{q-1}\psi_t\,dxdt - \iint_{Q_T} \nabla u\cdot\nabla\psi\,dxdt + \int_\Omega u_0(x)^{q-1}\psi(x,0)dx = 0$$

for every $\psi \in E$.

In [31], it has been shown that if $u_0 \in L^\infty(\Omega)$, $u_0^n \in C_0^\infty(\Omega)$ with $\|u_0^n\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}$, satisfies $u_0^n \to u_0$ in $L^p(\Omega)$ for every $1 \le p < \infty$ as $n \to \infty$, then the standard regularizations of (2'); namely

$$(2',n,\varepsilon) \qquad \begin{cases} (u_n^{q-1})_t - \Delta u_n^\varepsilon = 0\ , \quad u_n^\varepsilon\big|_{\partial\Omega} = \varepsilon \\ u(x,0) = u_0^n(x) + \varepsilon \end{cases}$$

have $C^\infty(\bar{\Omega}_T)$ solutions u_n^ε converge uniformly on compact subsets of Ω_T to a weak solution u of (2') as $n \to \infty$ and $\varepsilon \to 0$ simultanously and u lies in $C(\Omega_T) \cap L^\infty(\Omega_T)$.

(II) Semi-group Solution:

Definition: Let $f(x) \in L^1(\Omega)$, we define $u \in D(-\Delta)$ and $-\Delta u = f$ if $u \in \mathring{W}_1^1(\Omega)$ and $\int_\Omega \nabla u \cdot \nabla \phi\, dx = \int_\Omega f\phi\, dx \ \forall\ \phi \in \mathring{W}_\infty^1(\Omega)$.

Let $\phi(v)$ be any maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ which contains the origin and $f \in L^1(\Omega)$. Then $v \in D(-\Delta\phi)$ and $-\Delta\phi(v) \ni f$ if $v \in L^1(\Omega)$ and there is a $u \in D(-\Delta)$ such that $\phi(v(x)) \ni u(x)$ a.e. $x \in \Omega$ and $f = -\Delta u$.

From the paper by Brezis and Strauss [16], we know that $-\Delta\phi$ is an m-accretive operator in $L^1(\Omega)$. Then by the Crandall-Liggett Convergence Theorem, there exists a semi-group solution $v(\cdot,t) = \lim\limits_{n\to\infty}(I - t/n\ \Delta\phi)^{-1}v_0$ of $v_t - \Delta\phi(v) \ni 0$, $v(x,0) = v_0(x) \in \overline{D(-\Delta\phi)}$ and v lies in $C([0,T]|L^1(\Omega))$. Furthermore, we call v a strong solution of $v_t - \phi(v) \ni 0$ if v lies in $AC_{loc}(0,T|L^1(\Omega))$ and is differentiable a.e. [t] into $L^1(\Omega)$ such that $v_t(t) - \Delta\phi(v(t)) \ni 0$ a.e. [t].

In [23], Evans has investigated the differentiability of this semi-group solution under certain assumptions on ϕ. We summarize this result and some other standard results in the following theorem.

Theorem I : Let $\phi|\mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing function with $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$. Let ϕ^{-1} be uniformly Lipschitz continuous. Then for every $v_0 \in L^1(\Omega)$, the corresponding semi-group solution of $v_t - \Delta\phi(v) = 0$, $v(\cdot,t)$ is a strong solution and it satisfies:

(i) $v(x,0) = v_0(x)$

(ii) $v_t \in L^2_{loc}(0,T|L^2(\Omega))$

(iii) $\phi(v(\cdot,t)) \in W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega)$ a.e. [t]

(iv) $v_t - \Delta\phi(v) = 0$ a.e. [t]

Thus, v is differentiable almost everywhere into $L^2(\Omega)$ and so is also differentiable almost everywhere into $L^1(\Omega)$. Furthermore $D(-\Delta\phi)$ is dense in $L^1(\Omega)$ and we have the estimate:

(v) $\|v(\cdot,t)\|_{L^p(\Omega)} \le \|v_0\|_{L^p(\Omega)}$ for every $1 \le p \le \infty$ with $v_0 \in L^p(\Omega)$.

More details concerning m-accretive operators in Banach spaces, nonlinear semi-groups, strong solutions and their differentiability, one can refer to reference like [6] and [10]. A good simple survey can be found in [21] and [24].

SECTION II. PROOF OF LEMMA I

Lemma I: Let $2 < q$ if $N \le 2$ or $2 < q < \frac{2N}{N-2}$ if $N > 2$. then, there exists $C = C(q,\Omega)$ such that if $v = u^{q-1}$ is the semi-group solution of (2) and T^* is its extinction time, we have,

$$(T^* - t)^{\frac{1}{q-2}} \le C(q,\Omega)\Big(\int_\Omega u^q(x,t)\,dx\Big)^{1/q} .$$

Proof:

(i) Since $v_0 = u_0^{q-1}$ is in $L^\infty(\Omega)$, Evan's result in [23] can be applied. Thus $v(x,t)$ is a strong solution and,

$$v_t \in L^2{}_{loc}(0,T|L^2(\Omega))$$

$$v^{\frac{1}{q-1}}(\cdot,t) \in W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega) \quad \text{a.e. [t]}$$

$$\Delta v^{\frac{1}{q-1}} = v_t \quad \text{a.e. [t].}$$

and,

$$\|u(t)^{q-1}\|_{L^p(\Omega)} \le \|u_0{}^{q-1}\|_{L^p(\Omega)} \quad \text{for every } 1 \le p \le \infty .$$

Let us recall that if $m > 1$, $w^m(t) \in C([0,T]|L^1(\Omega))$, w^{m-1} and w_t lie respectively in $L^p(Q_T)$ and $L^{p^*}(Q_T)$ where $p > 1$ (p^* being the Hölder conjugate of p), it is not difficult to see that $\int_\Omega w^m(x,t)dx$ lies in $AC_{loc}(0,T)$ and, $\frac{d}{dt}\int_\Omega w^m(x,t)dx = m\int_\Omega w^{m-1}(x,t)w_t(x,t)dx$ a.e. [t].

Hence, in particular, if we let $f(t) = \int_\Omega v(x,t)^{\frac{q}{q-1}}dx$, $f(t)$ is locally absolutely continuous and $f'(t) = (\frac{q}{q-1})\int_\Omega v^{\frac{1}{q-1}}(x,t)v_t(x,t)dx$ a.e. [t].

(ii) We note that $(f(t))^{\frac{q-2}{q}} = (\int_\Omega v^{\frac{1}{q-1}}(x,t)dx)^{\frac{q-2}{q}}$ is also locally absolutely continuous and

$$\frac{d}{dt}(f(t))^{\frac{q-2}{q}} = \frac{(\frac{q-2}{q-1})\int_\Omega v^{\frac{1}{q-1}}(x,t)v_t(x,t)dx}{(\int_\Omega v^{\frac{q}{q-1}}(x,t)dx)^{2/q}} \quad \text{a.e. [t]}$$

so long as $f(t) > 0$.

(iii) By the fact that $v^{\frac{1}{q-1}}(\cdot,t) \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ and $\Delta v^{\frac{1}{q-1}}(x,t) = v_t(x,t)$ a.e. [t], we have,

$$\int_\Omega v^{\frac{1}{q-1}}(x,t)v_t(x,t)dx = -\int_\Omega |\nabla v^{\frac{1}{q-1}}(x,t)|^2dx$$

$$= (\frac{q-1}{q})\frac{d}{dt}\int_\Omega v^{\frac{q}{q-1}}(x,t)dx \quad \text{by (i)}$$

But,

$$-\int_\Omega |\nabla v^{\frac{1}{q-1}}(x,t)|^2dx \le -C(q,\Omega)(\int_\Omega v^{\frac{q}{q-1}}(x,t)dx)^{2/q}$$

by Sobolev's Lemma and the restriction on q. Thus,

$$\frac{d}{dt}(\int_\Omega v^{\frac{q}{q-1}}(x,t)dx)^{\frac{q-2}{q}} = \frac{d}{dt}(f(t))^{\frac{q-2}{q}} \le -C(q,\Omega)$$

a.e. [t] so long as $f(t) > 0$. The existence of an extinction time T^* is immediate. Furthermore, let $t < t' < T^*$ and integrate from t to t' to find

$$\left(\int_\Omega v^{\frac{q}{q-1}}(x,t')dx\right)^{\frac{q-2}{q}} - \left(\int_\Omega v^{\frac{q}{q-1}}(x,t)dx\right)^{\frac{q-2}{q}} \le C(q,\Omega)(t' - t)$$

(iv) Finally by Theorem I again, since

$$v(\cdot,t) \in C(0,T|L^1(\Omega)) \quad \text{and} \quad \|v(\cdot,t)\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)} ,$$

a simple interpolation shows $v(\cdot,t)$ lies in $C(0,T|L^p(\Omega))\ \forall\ 1 \le p < \infty$. Now let $t' \to T^*$, to deduce,

$$\left(\int_\Omega v^{\frac{q}{q-1}}(x,t)dx\right)^{\frac{q-2}{q}} \ge C(q,\Omega)(T^* - t)$$

<u>Proposition I.1</u>: Let q be in the same ranges as in Lemma 1 and $\beta(s)$ be such that $\beta(0) = 0$, $\beta'(0) = 0$ and $\beta'(s) > 0$ for $s > 0$. Furthermore, suppose there exists $M > m > 0$ such that $ms^{q-2} \le \beta'(s) \le Ms^{q-2}$, let $u(x,t)$ be the corresponding solution of,

$$\begin{cases} \beta'(u)u_t - \Delta u = 0 , & u\big|_{\partial\Omega} = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Then the estimate

$$\left(\int_\Omega u^q(x,t)dx\right)^{\frac{q-2}{q}} \ge C(q,\Omega)(T^* - t)$$

or

$$\left(\int_\Omega \beta(u(x,t))^{\frac{q}{q-1}}dx\right)^{\frac{q-2}{q}} \ge C(q,\Omega)(T^* - t)$$

remains valid.

<u>Remark</u>: This proposition is interesting when one wants to generalize the results of this paper to other nonlinearities $\beta(s)$, and this will be used in Section VI later. Of course, we should always bear in mind that we assume our initial data to be non-negative, not identically zero and is in $L^\infty(\Omega)$ unless otherwise stated.

SECTION III. LEMMA II and LEMMA III

In this Section, we will use the regularization $(2',n,\varepsilon)$ mentioned in section (I), namely,

$$(2',n,\varepsilon) \qquad \begin{cases} ({u_n^{\varepsilon}}^{q-1})_t - \Delta u_n^{\varepsilon} = 0, \quad u_n^{\varepsilon}\big|_{\partial\Omega} = \varepsilon \\ u_n^{\varepsilon}(x,0) = u_0^n(x) + \varepsilon \end{cases}$$

We will prove two other lemmas. These lemmas together with Lemma I, will ultimately lead to the stability results we want.

Furthermore, according to Evan's result, if $v(t) = u(t)^{q-1}$ is the semi-group solution of (2) with $u_0 \in L^{\infty}(\Omega)$, then $u(t) \in \mathring{W}_2^1(\Omega) \cap W_2^2(\Omega)$ a.e. [t]. Hence, to consider $L^{\infty}(\Omega)$ initial data, it is sufficient to assume $u_0 \in \mathring{W}_2^1(\Omega) \cap L^{\infty}(\Omega)$. So here we assume that $u_0 \in \mathring{W}_2^1(\Omega) \cap L^{\infty}(\Omega)$ and $u_0^n \in C_0^{\infty}(\Omega)$, $u_0^n \to u_0$ in $\mathring{W}_2^1(\Omega)$ and $u_0^n \to u_0$ in $L^p(\Omega)$, $1 \le p < \infty$.

<u>Lemma II</u>: Let $u(x,t)$ be the solution of (2') with $u_0 \in \mathring{W}_2^1(\Omega) \cap L^{\infty}(\Omega)$. Then,

$$\left(\int_\Omega u^q(x,t)dx\right)^{\frac{q-2}{q}} \le (q-1)\frac{\int_\Omega |\nabla u_0|^2}{\left(\int_\Omega u_0^q\right)^{2/q}}(T^* - t) \quad \forall t \le T^*$$

where T^* is the extinction time.

<u>Proof</u>:

(i) Let u_n^{ε} be the solution of $(2',n,\varepsilon)$, $u_n^{\varepsilon} \in C^{\infty}(\bar{Q}_{T^*})$ and that

$$\frac{\int_\Omega |\nabla u_n^{\varepsilon}(x,t)|^2 dx}{\left(\int_\Omega {u_n^{\varepsilon}}^q(x,t)dx\right)^{2/q}}$$

is a decreasing function of t. Indeed, using the fact that,

$$\int_\Omega ({u_n^{\varepsilon}}^{q-1})_t u_n^{\varepsilon} dx = \int_\Omega u_n^{\varepsilon}\Delta u_n^{\varepsilon} dx = -\int_\Omega |\nabla u_n^{\varepsilon}|^2 dx + \int_\Omega u_n^{\varepsilon}\frac{du_n^{\varepsilon}}{d\eta}dx \le 0$$

where η is the outward normal at $\partial\Omega$, we have,

$$\frac{d}{dt}\frac{\int_\Omega |\nabla u_n^\varepsilon|^2}{(\int_\Omega u_n^{\varepsilon^q})^{2/q}} = \frac{(\int_\Omega u_n^{\varepsilon^q})^{2/q}(2\int_\Omega \nabla u_n^\varepsilon\cdot\nabla u_{n_t}^\varepsilon) - (\int_\Omega |\nabla u_n^\varepsilon|^2)\frac{d}{dt}(\int_\Omega u_n^{\varepsilon^q})^{2/q}}{(\int_\Omega u_n^{\varepsilon^q})^{4/q}}$$

$$= \frac{-2(\int_\Omega u_n^{\varepsilon^q})(\int_\Omega u_{n_t}^\varepsilon \Delta u_n^\varepsilon) + 2(\int_\Omega u_n^{\varepsilon^{q-1}} u_{n_t}^\varepsilon)(\int_\Omega u_n^\varepsilon \Delta u_n^\varepsilon - \int_{\partial\Omega} u_n^\varepsilon \frac{du_n^\varepsilon}{dn})}{(\int_\Omega u_n^{\varepsilon^q})^{\frac{q+2}{q}}}$$

$$< \frac{2[(\int_\Omega u_n^{\varepsilon^{q-1}} u_{n_t}^\varepsilon)(\int_\Omega u_n^\varepsilon \Delta u_n^\varepsilon) - (\int_\Omega u_n^{\varepsilon^q})(\int_\Omega u_{n_t}^\varepsilon \Delta u_n^\varepsilon)]}{(\int_\Omega u_n^{\varepsilon^q})^{\frac{q+2}{q}}} < 0$$

by Hölder's Inequality.

(ii) Next, we note that,

$$-\frac{d}{dt}\left(\int_\Omega (u_n^\varepsilon - \varepsilon)^q\right)^{\frac{q-2}{q}} < (q-1)\frac{\int_\Omega |\nabla u_n^\varepsilon|^2}{(\int_\Omega (u_n^\varepsilon - \varepsilon)^q)^{2/q}}$$

for,

$$-\frac{d}{dt}\left(\int_\Omega (u_n^\varepsilon - \varepsilon)^q\right)^{\frac{q-2}{q}} = \left(\frac{q-2}{q-1}\right)\frac{-\int_\Omega (1-\frac{\varepsilon}{u_n^\varepsilon})^{q-2}(u_n^\varepsilon - \varepsilon)\Delta u_n^\varepsilon}{(\int_\Omega (u_n^\varepsilon - \varepsilon)^q)^{2/q}}$$

and,

$$-\int_\Omega \left(1 - \frac{\varepsilon}{u_n^\varepsilon}\right)^{q-2}(u_n^\varepsilon - \varepsilon)\Delta u_n^\varepsilon = -\int_\Omega \frac{(u_n^\varepsilon - \varepsilon)^{q-1}}{u_n^\varepsilon}\Delta u_n^\varepsilon = \int_\Omega \nabla\left[\frac{(u_n^\varepsilon - \varepsilon)^{q-1}}{u_n^\varepsilon}\right]\cdot\nabla u_n^\varepsilon$$

$$= \int_\Omega \left[\left(\frac{u_n^\varepsilon - \varepsilon}{u_n^\varepsilon}\right)^{q-2}\nabla u_n^\varepsilon + (q-2)\left(\frac{u_n^\varepsilon - \varepsilon}{u_n^\varepsilon}\right)^{q-2}\left(\frac{\varepsilon}{u_n^\varepsilon}\right)\nabla u_n^\varepsilon\right]\cdot\nabla u_n^\varepsilon$$

$$< \int_\Omega |\nabla u_n^\varepsilon|^2 + \int_\Omega (q-2)|\nabla u_n^\varepsilon|^2 = (q-1)\int_\Omega |\nabla u_n^\varepsilon|^2$$

Since $u_n^\varepsilon > \varepsilon$ by the Maximum Principle. Hence,

$$-\frac{d}{dt}\left(\int_\Omega (u_n^\varepsilon - \varepsilon)^q\right)^{\frac{q-2}{q}} < (q-1)\frac{\int_\Omega |\nabla u_n^\varepsilon|_2}{\left(\int_\Omega u_n^{\varepsilon q}\right)^{2/q}} \cdot \frac{\left(\int_\Omega u_n^{\varepsilon q}\right)^{2/q}}{\left(\int_\Omega (u_n^\varepsilon - \varepsilon)^q\right)^{2/q}}$$

(iii) Let $[t_1,t_2] \subset (0,T^*]$. By the results we will prove in section (IV), we have, $\int_\Omega u_n^\varepsilon(t)^q \to \int_\Omega u^q(t)$ uniformly on $[t_1, t_2]$. Thus,

$$-\frac{d}{dt}\left(\int_\Omega (u_n^\varepsilon - \varepsilon)^q\right)^{\frac{q-2}{q}} < (q-1)\frac{\int_\Omega |\nabla u_0^n|^2}{\left(\int_\Omega u_0^{n\,q}\right)^{2/q}}(1 + \delta(\varepsilon,n,t_1,t_2))$$

where $\delta(\varepsilon,n,t_1,t_2) \to 0$ as $\varepsilon \to 0$, $n \to \infty$ and so,

$$\left(\int_\Omega (u_n^\varepsilon(t_1) - \varepsilon)^q\right)^{\frac{q-2}{q}} - \left(\int_\Omega (u_n^\varepsilon(t_2) - \varepsilon)^q\right)^{\frac{q-2}{q}}$$

$$< \frac{\int_\Omega |\nabla u_0^n|^2}{\left(\int_\Omega u_0^{n\,q}\right)^{2/q}}(t_2 - t_1)(q-1)(1 + \delta(\varepsilon,n,t_1,t_2))$$

Finally, let $\varepsilon \to 0$, $n \to \infty$, $t_1 = t$ and $t_2 \to T^*$, we are done.

<u>Remark:</u>

In Proposition II.1, we mentioned that the estimate given by Lemma I is also true for the equation:

$$\begin{cases} \beta'(u)u_t - \Delta u = 0, \ u\big|_{\partial\Omega} = 0 \\ u(x,0) = u_0(x) \end{cases}$$

where $\beta(s)$ is such that $\beta(0) = 0$, $\beta'(0) = 0$, $\beta'(s) > 0$ for $s > 0$ and there exists $M > m > 0$ such that $ms^{q-2} < \beta'(s) < Ms^{q-2}$. It turns out that the previous estimate can not be generalized easily, but this is also true provided that $\beta'(s)$ tends to

$(q-1)s^{q-2}$ "sufficiently fast" as $s \to 0$. Of course, we will need a much more complicated analysis and this will be made clear in Section VI.

<u>Fundamental Lemma III</u>: Let us introduce the coordinate transformation $t = T^*(1 - e^{-(q-2)T})$ and define:

$$W(x,T) = \frac{u(x,T^*(1 - e^{-(q-2)T}))e^{T}}{((q-2)T^*)^{\frac{1}{q-2}}},$$

then W satisfies:

$$\Delta W + (q-1)W^{q-1} = (q-1)W^{q-2}W_T$$

and there exists,

$$\{T_n\}, \quad T_n \to \infty \quad \text{such that} \quad \|W^{q-2}W_T(T_n)\|_{L^{\frac{q}{q-1}}(\Omega)} \to 0$$

<u>Proof</u>:

We consider the corresponding transformation of $u_n^{\varepsilon}(x,t)$,

$$W_n^{\varepsilon}(x,t) = \frac{u_n^{\varepsilon}(x,T^*(1 - e^{-(q-2)T}))e^{T}}{((q-2)T^*)^{\frac{1}{q-2}}},$$

which satisfies,

$$\begin{cases} \Delta W_n^{\varepsilon}(x,T) + (q-1)W_n^{\varepsilon}(x,T)^{q-1} = (q-1)W_n^{\varepsilon}(x,T)^{q-2}W_{n_T}^{\varepsilon}(x,T) \\[2ex] W_n^{\varepsilon}(x,0) = \dfrac{u_0^n(x)+\varepsilon}{((q-2)T^*)^{\frac{q}{q-2}}}, \quad W_n^{\varepsilon}\Big|_{\partial\Omega} = \dfrac{\varepsilon e^{T}}{((q-2)T^*)^{\frac{1}{q-2}}} \end{cases}$$

The proof of the lemma is long and so we divide it into five steps.

(i) By direct differentiation

$$\frac{d}{dT}\left[\frac{1}{2}\int_\Omega |\nabla W_n^\varepsilon|^2 dx - \left(\frac{q-1}{q}\right)\int_\Omega W_n^{\varepsilon^q} dx\right]$$

$$= -(q-1)\int_\Omega W_n^{\varepsilon^{q-2}} |W_{n_T}^\varepsilon|^2 dx + \int_{\partial\Omega} \frac{\varepsilon e^T}{((q-2)T^*)^{\frac{1}{q-2}}} \frac{dW_n^\varepsilon}{d\eta} dx \leq 0$$

hence,

$$\frac{1}{2}\int_\Omega |\nabla W_n^\varepsilon|^2 dx - \left(\frac{q-1}{q}\right)\int_\Omega W_n^{\varepsilon^q} dx$$

$$\leq \frac{1}{2}\int_\Omega \frac{|u_0^n|^2}{((q-2))T^{*\frac{2}{q-2}}} dx - \left(\frac{q-1}{q}\right)\int_\Omega \frac{u_0^n(x)^q}{((q-2)T^*)^{\frac{q}{q-2}}} dx$$

An immediate consequence of this inequality is,

$$\frac{1}{2}\int_\Omega |\nabla W|^2 dx \leq \frac{1}{2}\int_\Omega \frac{|\nabla u_0|^2}{((q-2)T^*)^{\frac{2}{q-2}}} dx - \left(\frac{q-1}{q}\right)\int_\Omega \frac{u_0^q}{((q-2)T^*)^{\frac{q}{q-2}}} dx + \left(\frac{q-1}{q}\right)\int_\Omega W^q dx \leq M$$

uniformly in T where M is independent of T. This is due to the uniform boundedness of $\int_\Omega W^q dx$ implied by Lemma II and the lower semi-continuity property of the norm of a Banach space with respect to weak convergence.

(ii) Apply Hölder's inequality to find,

$$\int_\Omega |W_n^{\varepsilon^{q-2}}(x,T) W_{n_T}^\varepsilon(x,T)|^{\frac{q}{q-1}} dx \leq \left(\int_\Omega W_n^{\varepsilon^q}(x,T)dx\right)^{\frac{q-2}{2(q-1)}} \left[\int_\Omega W_n^{\varepsilon^{q-2}}(T) W_{n_T}^\varepsilon(T)dx\right]^{\frac{q}{2(q-1)}}$$

The uniform convergence of u_n^ε in $L^q(\Omega)$ to u on compact subinterval of $(0,T^*]$ implies the uniform convergence of W_n^ε in $L^q(\Omega)$ to W on $a \leq T \leq b$ where $0 < a < b < \infty$. But $\int_\Omega W^q(x,T)dx$ is uniformly bounded in T, hence, for $\tau \in [a,b]$ as $\varepsilon \to 0$,

$$\int_\Omega |W_n^{\varepsilon^{q-2}}(x,r)W^{\varepsilon}_{n_T}(x,r)|^{\frac{q}{q-1}}dx \le (M + \delta(a,b,n,\varepsilon))^{\frac{q-2}{2(q-1)}}\Big(\int_\Omega W_n^{\varepsilon^{q-2}}(r)W^{\varepsilon}_{n_T}(r)dx\Big)^{\frac{q}{2(q-1)}}$$

where $\delta(a,b,n,\varepsilon) \to 0$ as $\varepsilon \to 0$, $n \to \infty$. Furthermore,

$$-\frac{d}{dT}\Big[\frac{1}{2}\int_\Omega |\nabla W_n^{\varepsilon}(x,r)|^2dx - \Big(\frac{q-1}{q}\Big)\int_\Omega W_n^{\varepsilon^{q}}(x,r)dx\Big]$$

$$\ge \int_\Omega W_n^{\varepsilon^{q-2}}(x,r)W^{\varepsilon}_{n_T}(x,r)^2\,dx \ge 0$$

and therefore,

$$\int_\Omega |W_n^{\varepsilon^{q-1}}(x,r)W^{\varepsilon}_{n_T}(x,r)|^{\frac{q}{q-1}}dx$$

$$\le (M + \delta(a,b,n,\varepsilon))^{\frac{q-2}{2(q-1)}}\Big\{\frac{-d}{dT}\Big[\frac{1}{2}\int_\Omega |\nabla W_n^{\varepsilon}(r)|^2dx - \Big(\frac{q-1}{q}\Big)\int_\Omega W_n^{\varepsilon^{q}}(r)dx\Big]\Big\}^{\frac{q}{2(q-1)}}$$

which implies,

$$\Big[\Big(\int_\Omega |W_n^{\varepsilon^{q-2}}(x,r)W^{\varepsilon}_{n_T}(x,r)|^{\frac{q}{q-1}}dx\Big)^{\frac{q-1}{q}}\Big]^2$$

$$\le (M + \delta(a,b,n,\varepsilon))\Big(\frac{-d}{dT}\Big[\frac{1}{2}\int_\Omega |\nabla W_n^{\varepsilon}(r)|^2dx - \Big(\frac{q-1}{q}\Big)\int_\Omega W_n^{\varepsilon^{q}}(r)dx\Big]\Big)$$

By the above estimate,

$$\int_a^b \Big(\|W_n^{\varepsilon^{q-2}}W^{\varepsilon}_{n_T}\|_{L^{\frac{q}{q-1}}(\Omega)}\Big)^2 \le (M + \delta(a,b,n,\varepsilon))^{\frac{q-2}{q}}\,[F(W_n^{\varepsilon}(a)) - F(W_n^{\varepsilon}(b))]$$

$$\le (M + \delta(a,b,n,\varepsilon))^{\frac{q-2}{q}}\Big[F\Big(\frac{u_0^n}{((q-2)T^*)^{\frac{1}{q-2}}}\Big) - F(W_n^{\varepsilon}(b))\Big]$$

Since by (i), $F(W_n^{\varepsilon}(T))$ is monotonic decreasing where F is the nonlinear functional defined by,

$$F(h) = \int_\Omega \left(\frac{1}{2}|\nabla h|^2 - \left(\frac{q-1}{q}\right)h^q\right)dx, \quad h \in \mathring{W}_2^1(\Omega) \quad L^q(\Omega) .$$

Again from (i)

$$F(W_n^\varepsilon(T)) \le F\left(\frac{u_0^n}{((q-2)T^*)^{\frac{1}{q-2}}}\right)$$

Thus, the right hand side of the inequality is uniformly bounded with respect to n and ε which in turns implies $W_n^{\varepsilon^{q-2}} W_{n_T}^\varepsilon$ is uniformly bounded with respect to n and ε in $L^2\left(a,b|L^{\frac{q}{q-1}}(\Omega)\right)$ and so there exists a subsequence $\{\varepsilon_k\}$ such that $W_n^{\varepsilon_k^{q-2}} W_{n_T}^{\varepsilon_k} \rightharpoonup h$ for some $h \in L^2\left(a,b|L^{\frac{q}{q-1}}(\Omega)\right)$. For simplicity, we denote $\{\varepsilon_k\}$ by $\{\varepsilon\}$. (Let us recall that $\rightharpoonup$ stands for weak convergence in Banach spaces.)

(iii) We claim that in fact $h = W^{q-2}W_T$. To see this, we note that (i) together with the uniform boundedness of $\int_\Omega W^q dx$ and the uniform convergence of W_n^ε to W in $L^{\frac{q}{q-1}}(\Omega)$, we have

$$\int_\Omega |\nabla W_n^\varepsilon(x,T)|^2 dx \le C(s,t) \quad \text{on} \quad s \le T \le t$$

$$\int_s^t \int_\Omega |\nabla W_n^\varepsilon(x,T)|^2 dx dT \le C(s,t)(s-t)$$

Thus, $W_n^\varepsilon(x,T) - \dfrac{\varepsilon e^T}{((q-2)T^*)^{\frac{1}{q-2}}}$ is uniformly bounded in $L^2(a,b|\mathring{W}_2^1(\Omega))$ and it follows that, $V_n^\varepsilon = W_n^\varepsilon(x,T) - \dfrac{\varepsilon e^T}{((q-2)T^*)^{\frac{1}{q-2}}} \rightharpoonup W$ in $L^2(s,t|\mathring{W}_2^1(\Omega))$.

(iv) Hence, for $\rho \in C_0^\infty(\Omega \times (s,t))$

$$\lim_{\substack{\varepsilon \to 0 \\ n \to \infty}} \int_s^t \int_\Omega W_n^{\varepsilon^{q-2}} W_{n_T}^\varepsilon \rho dx dT = -\left(\frac{1}{q-1}\right) \int_s^t \int_\Omega \nabla W \cdot \nabla \rho dx dT + \int_s^t \int_\Omega W^{q-1} \rho dx dT .$$

On the other hand,

$$\Delta W + (q-1)W^{q-1} = (q-1)W^{q-2}W_T \quad \text{a.e.} \quad [T]$$

$$W_T \in L^2_{loc}(s,t|L^2(\Omega)) \quad \text{and} \quad W(T) \in \mathring{W}^1_2(\Omega) \quad W^2_2(\Omega) .$$

Thus,

$$-\left(\frac{1}{q-1}\right)\int_a^b\int_\Omega \nabla W\cdot\nabla\rho\, dx dT + \int_a^b\int_\Omega W^{q-1}\rho\, dx dT$$

$$= \left(\frac{1}{q-1}\right)\int_a^b\int_\Omega (\Delta W + W^{q-1})\rho\, dx dT = \int_a^b\int_\Omega \rho W^{q-2}W_T\, dx dT .$$

But $C_0^\infty(\Omega \times (a,b))$ is dense in $L^2(a,b|L^q(\Omega))$ which is the dual of $L^2(a,b|L^{\frac{q}{q-1}}(\Omega))$.

Therefore,

$$W_n^{\varepsilon^{q-1}} W^\varepsilon_{n_T} \longrightarrow W^{q-2}W_T \quad \text{in} \quad L^2(a,b|L^{\frac{q}{q-1}}(\Omega)) .$$

(v) Finally, by the lower semi-continuous property of the norm of a Banach space with respect to weak convergence, we have,

$$\int_a^b \left[\left(\int_\Omega |W^{q-2}W_T|^{\frac{q}{q-1}}dx\right)^{\frac{q-1}{q}}\right]^2 dT \leq \varliminf_{\substack{\varepsilon\to 0\\ n\to\infty}} \int_a^b \left[\left(\int_\Omega |W_{n_T}^{\varepsilon^{q-2}} W^\varepsilon_{n_T}|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}\right]^2 dT$$

$$\leq \overline{\lim_{\substack{\varepsilon\to 0\\ n\to\infty}}}\, (M + \delta(a,b,n,\varepsilon))^{\frac{q-2}{q}}\left[F\left(\frac{u_o(x)}{((q-2)T^*)^{\frac{1}{q-2}}}\right) - F(W^\varepsilon_n(b))\right]$$

But,

$$-\varliminf_{\substack{\varepsilon\to 0\\ n\to\infty}} \int_\Omega |\nabla W^\varepsilon_n|^2 dx \leq -\int_\Omega |\nabla W(b)|^2 dx ,$$

hence,

$$-\varliminf_{\substack{\varepsilon\to 0\\ n\to\infty}} F(W^\varepsilon_n(b)) \leq -F(W(b))$$

which implies,

$$\int_a^b \left(\|W^{q-2}W_T\|_{L^{\frac{q}{q-1}}(\Omega)}\right)^2 dT \le M^{\frac{q-2}{q}}\left[F\left(\frac{u_o(x)}{((q-2)T^*)^{\frac{1}{q-2}}}\right) - F(W(b))\right]$$

where M is independent of a and b. Furthermore $|F(W(b))|$ is uniformly bounded by (i) and the second lemma. Thus, there exists a sequence $\{T_n\}$, $T_n \to \infty$ such that $\|W^{q-2}W_T(T_n)\|_{L^{\frac{q}{q-1}}(\Omega)} \to 0$ as $T_n \to \infty$.

Now, let us restate the main theorem of this paper and prove it. The proof is basically the same as in the paper of Holland and Berryman [12], since it is not long, for the sake of completeness we reproduce it here.

Main Theorem: Let $p(x,t) = u(x,t)/[(q-2)(T^* - t)]^{\frac{q}{q-2}}$, then

(1) If $2 < q$ for $N \le 2$ or $2 < q < 2N/(N-2)$ for $N > 2$, there exists an increasing sequence of times $t_n \to T^*$ and $p(\cdot,t_n) \to S(\cdot)$ in $\overset{\circ}{W}{}^1_2(\Omega)$ where S is a positive classical solution of (3).

(2) If $q \ge 2N/(N-2)$, $N > 2$, then there is a non-negative weak solution $S(\cdot)$ of (3) and an increasing sequence of times $\{t_n\}$, $t_n \to T^*$ such that $p(\cdot,t_n) \to S(\cdot)$ in $\overset{\circ}{W}{}^1_2(\Omega)$.

Proof of (1):

(i) Let us first recall if we put $t_n = T^*(1 - e^{-(q-2)T_n})$ where $\{T_n\}$ is the sequence of the third fundamental lemma, then $p(\cdot,t_n)$ is transformed into $W(\cdot,T_n)$ where $W(\cdot,T_n)$ satisfies,

$$\Delta W + (q-1)W^{q-1} = (q-1)W^{q-2}W_T$$

Clearly,

$$\int_\Omega |\Delta W(x,T_n)|^{\frac{q}{q-1}}dx \le A\int_\Omega |W^{q-1}(x,T_n)|^{\frac{q}{q-1}}dx + B\int_\Omega |W^{q-2}(x,T_n)W_T(x,T_n)|^{\frac{q}{q-1}}dx$$

for some positive constants A,B. The first term on the right is bounded by the second fundamental lemma. The second term converges to zero by the third fundamental lemma.

Hence, $\|W(\cdot,T_n)\|_{W^2_{\frac{q}{q-1}}(\Omega)}$ is bounded by the L^p theory of the second order linear elliptic problem. (For the details of the L^p theory see [20], [27]). By the weak embedding property, there exists $\{T_n\}$, $T_n \to \infty$ such that $W(\cdot,T_n) \to S(\cdot)$ in $\mathring{W}^1_{\frac{q}{q-1}}(\Omega)$ as $T_n \to \infty$.

(ii) the convergence is actually better. First we recall that in (i) of the proof of the third fundamental lemma, we have shown the uniform boundedness of $\int_\Omega |\nabla W(x,T)|^2 dx$ which implies $S(\cdot)$ must be in $\mathring{W}^1_2(\Omega)$ and $W(\cdot,T_n)$ in fact converges weakly to $S(\cdot)$ in $\mathring{W}^1_2(\Omega)$.

(iii) Next we show that $S(\cdot)$ is a classical solution of (3) i.e., S satisfies; $\Delta S + (q-1)S^{q-1} = 0$ in Ω with $S|_{\partial\Omega} = 0$. Indeed, let $\rho \in C_0^\infty(\Omega)$, we use,

$$\int_\Omega [-\nabla\rho\cdot\nabla W + (q-1)\rho W^{q-1}]dx = \int_\Omega (q-1)\rho W^{q-2} W_T dx$$

and let $T_n \to \infty$ to find,

$$\int_\Omega [-\nabla\rho\cdot\nabla S + (q-1)\rho S^{q-1})dx = 0 \ ,$$

since the right hand side converges to zero by the third fundamental lemma. Hence $S(\cdot)$ is a weak solution of (3). But by a simple bootstrap argument, a weak solution of (3) is in fact a classical solution of (3) if $N \leq 2$ or $2 < q < 2N/(N-2)$ for $N > 2$ which is our case here.

(iv) $S(x) \geq 0$ and is not the zero solution by Lemma I. But any non-negative solution of (3) with $S|_{\partial\Omega} = 0$ must be positive everywhere inside Ω by the maximum principle, thus $S(x) > 0$ in Ω.

(v) Finally we show that $W(\cdot,T_n) \to S(\cdot)$ in $\mathring{W}^1_2(\Omega)$.

To see this we multiply $\Delta W + (q-1)W^{q-1} = (q-1)W^{q-2}W_T$ by W and integrate. We have,

$$\int_\Omega [-|\nabla W(x,T_n)|^2 + (q-1)W^q(x,T_n)]dx = \int_\Omega (q-1)W^{q-1}W_T(x,T_n)dx \ .$$

Again invoking the third fundamental lemma and letting $T_n \to \infty$, we have,

$$\left(\frac{q}{q-1}\right) \lim_{T_n \to \infty} \int_\Omega |\nabla W(x,T_n)|^2 dx = \lim_{T_n \to \infty} \int_\Omega W^q(x,T_n)dx = \int_\Omega S^q(x)dx$$

due to the fact that $\int_\Omega W^q(x,T_n)dx \to \int_\Omega S^q(x)dx$ by Kondrachov compactness theorem ([26], p. 42). On the other hand,

$$\left(\frac{1}{q-1}\right) \int_\Omega |\nabla S(x)|^2 dx = \int_\Omega S^q(x)dx$$

Thus,

$$\|W(\cdot,T_n)\|_{W_2^1(\Omega)} \to \|S(\cdot)\|_{W_2^1(\Omega)},$$

together with the weak convergence which we have established,

$$W(\cdot,T_n) \to S(\cdot) \quad \text{in} \quad \mathring{W}_2^1(\Omega).$$

Proof of (2):

It is similar to the proof of (1), except that we no longer have our estimate in the first fundamental lemma and also because of the range of q, we can only conclude that $S(\cdot)$ is a non-negative weak solution of (3). For the details, one can refer to the original proof in [12].

SECTION IV

In this section, we will prove the convergence of the u_n^ε of $(2',n,\varepsilon)$ to the solution $u(x,t)$ of (2') where $v = u^{q-1}$ is the semi-group solution of (2). The convergence is in the space $C((0,T]|L^p(\Omega))$, $1 \le p < \infty$. We recall that in [31], it has been proved that u_n^ε converges on compact subsets of Q_T to a weak solution $u \in C(Q_T)$ $L^\infty(Q_T)$ for bounded initial data u_0. Obviously u_n^ε converges in $C((0,T]|L^p(\Omega))$, $1 \le p < \infty$, to u, the only question is whether u^{q-1} is the semi-group solution. We will show that this is true by proving a uniqueness theorem of the weak solution. (Note that the semi-group solution is also a weak solution).

Before we go on, for the convenience of computations we introduce on other notion of weak solution which is equivalent to the one given in [31]. It is certainly not difficult to see this especially when u is in $C(Q_T) \cap L^\infty(Q_T)$

Definition: $u^{q-1} \in C((0,T]|L^1(\Omega)) \cap L^\infty(Q_T)$ is a weak solution of

$$(2') \qquad \begin{cases} (u^{q-1})_t - \Delta u = 0, & u|_{\partial\Omega} = 0 \\ u(x,0) = u_0(x), & u_0^{q-1} \in L^1(\Omega) \end{cases}$$

if it satisfies;

$$\int_\Omega u^{q-1}(T)\rho(T) = \int_\Omega u_0^{q-1}\rho(0) + \int_{Q_T} u^{q-1}\rho_t + u\Delta\rho$$

for every $\rho \in C^2(\bar{Q}_T)$ and $\rho|_{\partial\Omega\times[0,T]} = 0$

Definition: $\bar{u}^{q-1} \in C((0,T]|L^1(\Omega)) \cap L^\infty(Q_T)$ is a super solution of (2') if its satisfies the inequality;

$$\int_\Omega u^{q-1}(T)\rho(T) \geq \int_\Omega u_0^{q-1}\rho(0) + \int_{Q_T} u^{q-1}\rho_t + u\Delta\rho$$

for every nonnegative $\rho \in C^2(\bar{Q}_T)$ with $\rho|_{\partial\Omega} = 0$. Subsolution are similary defined with $\geq$ replaced by $\leq$.

Proposition IV-1: Let $\underline{u}$ and $\bar{u}$ be a subsolution and a supersolution respectively. Then $\underline{u} \leq \bar{u}$ a.e. in Q_T.

(i) $\int_\Omega \underline{u}^{q-1}(T)\rho(T) \leq \int_\Omega u_0^{q-1}\rho(0) + \int_{Q_T} \underline{u}^{q-1}\rho_t + \underline{u}\Delta\rho$

Proof: $\int_\Omega \bar{u}^{q-1}(T)\rho(T) \geq \int_\Omega u_0^{q-1}\rho(0) + \int_{Q_T} \bar{u}^{q-1}\rho_t + \bar{u}\Delta\rho$ whenever ρ is a test function. Subtraction yields,

$$\int_\Omega (\underline{u}^{q-1}(T) - \bar{u}^{q-1}(T))\rho(T) \leq \int_{Q_T} (\underline{u}^{q-1} - \bar{u}^{q-1})\rho_t + (\underline{u} - \bar{u})\Delta\rho$$

Let

$$a(x,t) = \frac{\underline{u}^{q-1}(x,t) - \bar{u}^{q-1}(x,t)}{\underline{u}(x,t) - \bar{u}(x,t)}$$

$a(x,t) \in L^\infty(Q_T)$ and $a(x,t) \geq 0$. Thus,

$$\int_\Omega (\underline{u}^{q-1}(T) - \bar{u}^{\,q-1}(T))\rho(T) \leq \int_\Omega (\underline{u} - \bar{u})(a\rho_t + \Delta\rho)$$

(ii) We choose $\chi \in C_0^\infty(\Omega)$, $0 \le \chi \le 1$ and $a_n \in C^\infty(\bar{Q}_T)$ such that $\frac{1}{n} \le a_n \le \|a\|_{L^\infty(Q_T)} + \frac{1}{n}$ and $\frac{a_{n-a}}{\sqrt{a_n}} \to 0$ in $L^2(Q_T)$ and we solve the backward problems

$$\begin{cases} a_n \rho_{n_t} + \Delta\rho_n = M\lambda\rho_n, \quad M = \|a\|_{L^\infty(Q_T)} + \frac{1}{n} \\ \rho_n\big|_{\partial\Omega} = 0 \\ \rho_n(T) = \chi \end{cases}$$

We observe that;

(1) $0 \le \rho_n \le e^{\lambda(t-T)}$ on $\bar{Q}_T$ by the Maximum Principle

(2) $\int_{Q_T} a_n \rho_{n_t}^2 \le C$ uniformly in n

(1) is trivial and to see (2), we multiply the equation by ρ_{n_t} and integrate;

$$\int_\Omega \int_t^T a_n \rho_{n_t}^2 - \int_t^T \int_\Omega \nabla\rho n \cdot \nabla\rho_{n_t} = \int_\Omega \int_t^T M\lambda\rho_n d\rho_n$$

$$\int_{\Omega\times[t,T]} a_n \rho_{n_t}^2 - \frac{1}{2}\int_\Omega |\nabla\rho_n(T)|^2 + \frac{1}{2}\int_\Omega |\nabla\rho_n(t)|^2 = \frac{M\lambda}{2}\int_\Omega \rho_n(T)^2 - \rho_n(t)^2$$

$$\int_{\Omega\times[t,T]} a_n \rho_{n_t}^2 + \frac{1}{2}\int_\Omega |\nabla\rho_n(t)|^2 + \frac{M\lambda}{2}\int_\Omega |\rho_n(t)|^2 = \frac{1}{2}\int_\Omega |\nabla\chi|^2 + \frac{M\lambda}{2}\int_\Omega \chi^2$$

for any $t > 0$.

(iii) Now that using ρ_n as a test function, we have;

$$\int_\Omega (\underline{u}^{q-1}(T) - \bar{u}^{q-1}(T))\chi \le \int_{Q_T} (\underline{u} - \bar{u})M\lambda\rho_n \le \int_{Q_T} (\underline{u} - \bar{u}) + M\lambda e^{\lambda(t-T)}$$

$$+ \int_{Q_T} (a - a_n)(\underline{u} - \bar{u})\rho_{n_t} \left(\|\underline{u}\|_{L^\infty(Q_T)} + \|\bar{u}\|_{L^\infty(Q_T)}\right) \sqrt{\int_{Q_T} \left|\frac{a - a_n}{\sqrt{a_n}}\right|^2} \sqrt{\int_{Q_T} a_n \rho_{n_t}^2}$$

Let $n \to \infty$ and $\lambda \to 0$,

$$\int_\Omega (\underline{u}^{q-1}(T) - \bar{u}^{q-1}(T))\chi \le 0 \ .$$

Equipped with the above proposition, the following theorem is now immediate.

Theorem IV-1: The weak solution of (2') is unique.

Remarks:

(i) The uniqueness theorem can be proved through the notion of distribution solution as in [15].

(ii) We note that if the initial data u_0 is not in $L^\infty(\Omega)$ but simply in $L^1(\Omega)$, we don't even know we have a weak solution. However, we still have a semi-group solution and there is a more general result concerning the continuous dependence which implies: If $\phi(s)$ and $\phi_\varepsilon(s)$ are continuous and nondecreasing on $\mathbb{R}$, $\phi_\varepsilon(0) = \phi(0) = 0$ and $\phi_\varepsilon(s) \to \phi(s)$, $v_0^\varepsilon \to v_0$ in $L^1(\Omega)$ as $\varepsilon \to 0$, then the corresponding semi-group solution v^ε of,

$$\begin{cases} v_t^\varepsilon - \Delta\phi_\varepsilon(v^\varepsilon) \quad 0, \quad v^\varepsilon\big|_{\partial\Omega} = 0 \\ v^\varepsilon(x,0) = v_0^\varepsilon(x) \end{cases}$$

converges in $C([0,T]|L^1(\Omega))$ to v, the semi-group solution of

$$\begin{cases} v_t - \Delta\phi(v) \quad 0, \quad v\big|_{\partial\Omega} = 0 \\ v(x,0) = v_0(x) \end{cases}$$

Hence, we can obtain the result of this section directly from the semigroup theory. (For the details, one can refer to [9]). Indeed, looking at the regularizations of (2'),

$$(2',n,\varepsilon) \begin{cases} (u_n^\varepsilon)_t^{q-1} - \Delta u_n^\varepsilon = 0, \quad u^\varepsilon\big|_{\partial\Omega} = \varepsilon \\ u_n^\varepsilon(x,0) = u_0^n(x) + \varepsilon \end{cases}$$

where $\varepsilon > 0$, $\varepsilon \to 0$, $u_0^n \to u_0$ in $L^p(\Omega)\ \forall\ 1 \le p < \infty$, $u_0^n \in C_0^\infty(\Omega)$ and $\|u_0^n\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}$

Let $\phi_\varepsilon(s) = (s + \varepsilon^{q-1})^{\frac{1}{q-1}} - \varepsilon$ and $v_n^\varepsilon = u_n^{\varepsilon^{q-1}} - \varepsilon^{q-1}$, then $u_n^\varepsilon = (v_n^\varepsilon + \varepsilon^{q-1})^{\frac{1}{q-1}}$, let $U_n^\varepsilon = u_n^\varepsilon - \varepsilon$ so that $U_n^\varepsilon = \phi_\varepsilon(v_n^\varepsilon)$. Alternatively, $v_n^\varepsilon = \beta_\varepsilon(U_n^\varepsilon)$ where $\beta_\varepsilon(s) = \phi_\varepsilon^{-1}(s)$. It is not difficult to see that U_n^ε satisfies the equivalent equation $(2',n,\varepsilon)'$ where,

$$(2',n,\varepsilon)' \quad \begin{cases} \beta_\varepsilon(U_n^\varepsilon)_t - \Delta U_n^\varepsilon = 0, \quad U_n^\varepsilon\big|_{\partial\Omega} = 0 \\ U_n^\varepsilon\ (x,0) = u_0^n(x) \end{cases}$$

On identifying naturally $v_n^\varepsilon = \beta_\varepsilon(U_n^\varepsilon)$ as the semi-group solution of $(2,n,\varepsilon)$ where, $\Phi_\varepsilon(s) \to \Phi(s)$, $v_n^\varepsilon(x,0) \to v_0(x)$ in $L^p(\Omega)$ $\forall\ 1 \le p < \infty$; as $\varepsilon \to 0$, $v_n^\varepsilon(x,t)$ converges in $C([0,T]|L^p(\Omega))$ $\forall\ 1 \le p \le \infty$ to $v(x,t)$, the semigroup solution of (2).

(iii) Furthermore, by the regularizing effect of the equation

$$(1) \quad \begin{cases} v_t - \Delta v^m = 0, \quad v\big|_{\partial\Omega} = 0 \quad \text{where} \quad m > 0 \\ v(x,0) = v_0(x) \end{cases}$$

for certain ranges of q, the solution of (2) $v(t) = u^{q-1}(t)$ of (2) is in $L^\infty(\Omega)$ for $t > 0$ and we still have a weak solution in Q_T.

Indeed, Gary Schroeder in [32], using the result by Benilan and Vernon in [11], he obtained for the solution v of equation (1) which includes (2) as a particular case.

<u>Theorem IV-2</u>: Let $t_0 > 0$, $t > t_0$, $p_0 \ge 1$, $p_0 > \frac{N}{2}(1 - m)$ and if $v(\cdot,t_0) \in p_0(\Omega)$, then the following estimate is valid.

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le \left[\frac{2p_0p}{Nm(\frac{2}{N}p_0 + m - 1)}\right]^{\frac{1}{\frac{2}{N}p_0+m-1}} \left[2^{\left(\frac{p}{p-1}\right)\left(\frac{1}{\frac{2}{N}p_0p+m-1}\right)}\right] \|v(\cdot,t_0)\|_{L^{p_0}(\Omega)}^{\frac{2p_0}{N(\frac{2}{N}p_0+m-1)}} \quad \forall\ p > 1$$

Hence, for $p_0 = 1$, $N > 2$ and $2 < q < \frac{2N-2}{N-2}$, $v(\cdot,t) \in L^\infty(\Omega)$ for $t < 0$. Using the evolutional property of the semi-group solution, we write $v(t) = S(t)v_0(x)$ as usual in the semi-group theory, we have

$$v(t + \tau) = S(t + \tau)v_0(x) = S(t)(S(\tau)v_0(x)) \ \forall\ t, \tau \ge 0 .$$

Thus, we have brought ourselves to the setting of $L^\infty(\Omega)$ initial data and we have a bounded continuous weak solution of (2'). For the case where $q \ge \frac{2N-2}{N-2}$, nothing about the existence of weak solutions (except the semi-group solution) has been known yet.

(iv) In fact, in [32] we have a more general result for the equation $\Delta\phi(v) = v_t$. We have exactly the same results as in (ii), the only conditions we have to impose on ϕ are

$\phi(0) = 0$ and $\phi'(s) \geq Cs^{m-1}$ for some positive constant C. Alternatively, if we let $\beta(s) = \phi^{-1}(s)$ and consider the equation $\beta'(u)u_t = \Delta u$, the conditions become $\beta(0) = 0$ and $\beta'(s) \leq C'\beta(s)^{1-m}$. We will use this result when we generalize our previous results in Section VI.

(v) The previous proof also works for the porous media equation, for the analogous version one can refer to [4].

(vi) Finally, we note that because of the regularizing effect, all the previous asymptotic behaviour results remain true and we summarize it in the following corollary.

Corollary IV-1: For $N > 2$ and $2 < q < \frac{2N-2}{N-2}$, all the previous asymptotic behaviour results remain true provided $u_0 \in L^{q-1}(\Omega)$ or $v_0 = u_0^{q-1} \in L^1(\Omega)$.

SECTION V. THE INTERIOR AND BOUNDARY REGULARITY OF THE SOLUTION

In this section, we will study the regularity of $u(x,t)$ inside the parabolic cylinder $\Omega_{T^*} = \Omega \times (0,T^*)$. We will first study the case where the initial data u_0 is in $L^\infty(\Omega)$ and then consider general $L^1(\Omega)$ initial data. The continuity up to the boundary $\partial\Omega$ is also studied under certain assumptions on the boundary of Ω, $\partial\Omega$. We will divide the sections in two parts as follows.

Part I

As before, we consider the regularizations

(2',n,ε) $$(2',n,\varepsilon)\begin{cases} (u_n^\varepsilon)_t^{q-1} - \Delta u_n^\varepsilon = 0, & u_n^\varepsilon\big|_{\partial\Omega} = \varepsilon \\ u_n^\varepsilon(x,0) = u_0^n(x) + \varepsilon\ , \end{cases}$$

where $\varepsilon > 0$, $\varepsilon \to 0$, $u_0^n \to u_0$ in $L^p(\Omega)$, $1 \leq p < \infty$, $u_0^n \in C_0^\infty(\Omega)$ and $\|u_0^n\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$. The u_n^ε converge uniformly on compact subsets on Ω_T. The proof is a direct application of the main theorem of [31] which we stated as follows. For the details, one may refer to [31].

Theorem V-1: Consider the equation,

$$\beta(u)_t = \Delta u + \bar{p}\cdot(\bar{\nabla}(\gamma(u))) + F(x,t,u)$$

Denote by ∂Q_T the boundary of the parabolic cylinder, i.e $\partial Q_T = \{(x,t) \in \bar{Q}_T | t = 0$ or $x \in \partial\Omega\}$. Furthermore;

(i) $\beta \in C^2(\mathbb{R})$, $\beta(0) = 0$, $0 < \beta' < \infty$

$$0 < \mu_1(\delta) < \beta'(s) < \hat{\mu}(\delta) \quad \text{for} \quad |s| < \delta > 0$$

(ii) $p \in C^{1,1}(Q_T)$, $\gamma \in C^2(\mathbb{R})$, $F \in C^1(Q_T \times R)$.

If a bounded classical solution $u(x,t)$ exists, i.e. $u \in C^{2,1}(Q_T)$ and satisfies the equation pointwise in Q_T. Let C_1 be a positive constant such that, $\|u\|_{L^\infty(Q_T)}$, $\|\beta(u)\|_{L^\infty(Q_T)}$, $\|F(\cdot,\cdot,u)\|_{L^\infty(Q_T)}$, $\|\gamma'(u)\|_{L^\infty(Q_T)}$ and $\|\bar{p}\|_{L^\infty(Q_T)}$ are all less than C_1. Then, if Q_T' is a subdomain of Q_T, the modulus of continuity of u in Q_T' depends only on the data n, C_1, $\mu(\cdot)$, $\hat{\mu}_1(\cdot)$ and the dist $(Q_T', \partial Q_T)$.

We now state the theorem about the interior regularity of the solution.

<u>Theorem V-2</u>: Let $u(x,t)$ be the solution of (2) where the initial data u_0 is in $L^\infty(\Omega)$. Then $u(x,t)$ is a positive classical solution of (2) in the parabolic cylinder Q_{T^*} where T^* is the extinction time. More precisely $u(x,t) > 0$ in Q_{T^*}, $u \in C^\infty(Q_{T^*})$ and satisfies $(q-1)u^{q-2}u_t - \Delta u = 0$. This theorem is an immediate consequence of the following lemma.

<u>Lemma V-1</u> Let Ω_T denotes $\{(x,t) \in Q_T | t=T\}$ and $u(x,t)$ be the bounded continuous weak solution of (2'). If $u(x,t)$ vanishes at $(x^*,T^*) \in Q_T$, then it vanishes everywhere on Ω_{T^*}. We will go ahead to prove theorem V-2 and postpone the proof of the lemma until later.

<u>Proof:</u>

Using Lemma V-1, since u is positive in Q_{T^*} and u_n^ε converge on compact subsets of Q_{T^*}, u_n^ε are bounded away from zero uniformly in ε and n on compact subsets of Q_{T^*}. By some standard results in [26], we can obtain uniform bounds on u_n^ε in $H^{2\ell+\alpha,\ell+\alpha/2}(\bar{Q}_T')$ where $\alpha > 0$, for any $\ell \in Z^+$ and $Q_T' \subset\subset Q_{T^*}$. Hence, the degeneracy has been handled and we have a positive and classical solution in Q_{T^*}.

We now relax the restriction from $u_0 \in L^\infty(\Omega)$ to $u_0 \in L^{q-1}(\Omega)$; i.e. $v_0 \in L^1(\Omega)$ and invoke the semi-group theory. We summarise the results in the following corollary.

Corollary V-1 Let $u_0 \in L^{q-1}(\Omega)$, alternatively $v_0 \in L^1(\Omega)$ with $N > 2$, $2 < q < (2N-2)/(N-2)$. Then there exists a positive classical solution of (2') in Q_T.

Proof:

The proof is based on Theorem IV-2. The main machinery is again the regularizing effect of the solution and we omit the details here.

We now come back to the proof of lemma V-1.

Proof:

This proof is a generalization of Sabanina's lemma in the R^1 case ([29]). For simplicity we only provide a proof for $\Omega \subset R^2$, the R^N case proceeds in exactly the same way. We will prove it by contradiction.

Indeed, let us assume the contrary. Then there exists a point (x^{**}, y^{**}, T^*) where u does not vanish and consequently we can find a region G in Q_T as shown in Figure I such that $u(x,t)$ vanishes at (x^*, y^*, T^*) but u is greater then zero on A_3. (Here, without loss of generality, we assume $y^* = y^{**} = 0$ and $x^{**} > x^* > 0$). We let (r,θ) denote the polar co-ordinates.

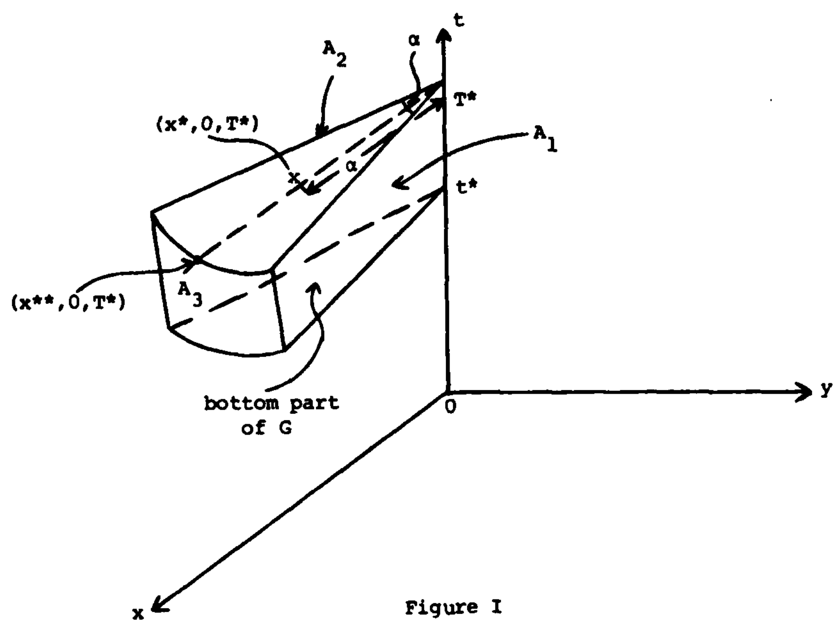


Figure I

We construct the comparison function, $w(x,y,t) = [m(t - t^*) + c]r^{\mu}\cos(\frac{\pi\theta}{2\alpha}) - M$ where $\mu > \pi/2\alpha$ and $0 < cR^{\mu} < M < m(T^* - t^*)d^{\mu}$. We define $Lu = (q - 1)u^{q-2}u_t - \Delta u$. Since $\Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}$, we have,

$\Delta\Lambda w = \Lambda\Delta w = \Lambda[m(t - t^*) + c][\mu^2 - (\frac{\pi}{2\alpha})^2]r^{\mu-2}\cos(\frac{\pi}{2\alpha}\theta)$ where $\Lambda > 0$ is to be chosen later. Thus,

$$L(\Lambda w) = ((q - 1)\Lambda^{q-2}w^{q-2})\Lambda w_t - \Lambda\Delta w$$

$$= ((q - 1)\Lambda^{q-2}w^{q-2})m\Lambda r^{\mu}\cos(\frac{\pi}{2\alpha}\theta)$$

$$- \Lambda[m(t - t^*) + c][\mu^2 - (\frac{\pi}{2\alpha})^2]r^{\mu-2}\cos(\frac{\pi}{2\alpha}\theta)$$

$$= \Lambda r^{\mu-2}\cos(\frac{\pi\theta}{2\alpha})\{(q - 1)\Lambda^{q-2}w^{q-2}mr^2 - [m(t - t^*) - c](\mu^2 - (\frac{\pi}{2\alpha})^2)\}$$

we now choose Λ sufficiently small so that $L(\Lambda w) < 0$ in G and $\Lambda|w| < \frac{1}{2}\inf_{(x,y,t)\in A_3} u(x,y,t)$ As a result, $L(\Lambda w) < 0$ in G, $\Lambda w|_{A_1} = \Lambda w|_{A_2} = -\Lambda M < 0$ and at the bottom part of G, $w = cr^{\mu}\cos(\frac{\pi\theta}{2\alpha}) - M < cR^{\mu} - M < 0$.

Now let $\{u_n^{\varepsilon}\}$ be the regularizations mentioned previously, u_n^{ε} converges uniformly on compact subsets of $\mathcal{D}_T$ to $u(x,y,t)$. Now, $u_n^{\varepsilon} > \frac{1}{2}\inf_{(x,y,t)\in A_3} u(x,y,t)$ holds in A_3 for small ε and large n. Hence, by the choice of β, $(\Lambda w - u_n^{\varepsilon})|_{A_3} < 0$. Thus, in G, we have $L\Lambda w - Lu_n^{\varepsilon} < 0$ with $\Lambda w - u_n^{\varepsilon} < 0$ along A_1, A_2, A_3 and bottom part of G. Finally we invoke the maximum principle of the linear case (see [28] for these standard techniques) to conclude that $\Lambda w < u_n^{\varepsilon}(x,y,t)$ in G.

In particular, $0 < \Lambda w(x^*,0,T^*) < u_n^{\varepsilon}(x^*,0,T^*)$. But $u_n^{\varepsilon}(x^*,0,T^*) \to u(x^*,0,T^*) = 0$ as $\varepsilon \to 0$, and $n \to \infty$. Hence, we have obtained a contradiction.

Finally, as in Proposition II-1, we have the following analogous proposition.

<u>Proposition V-1</u>: Let $\beta(s)$ be such that $\beta(0) = 0$, $\beta'(s) > 0$ for $s > 0$. Furthermore, suppose there exists $M > m > 0$ such that $ms^{q-2} < \beta'(s) < Ms^{q-2}$, let $u(x,t)$ be the corresponding solution of,

$$\begin{cases} \beta'(u)u_t - \Delta u = 0, & u\big|_{\partial\Omega} = 0 \\ u(x,0) = u_0(x) \end{cases}$$

with $u_0 \in L^\infty(\Omega)$, and $u_0 \geq 0$, $u_0 \not\equiv 0$. Then, u is a positive classical solution in Q_T. (The previous proof can be modified trivially to this general case.)

Remark: If u_0 is simply in $L^{q-1}(\Omega)$ for $N > 2$, where $2 < q < \frac{2N-2}{N-2}$, then there exists a positive classical solution in Q_T due to regularizing effect. (See Section IV for the details about the regularizing effect).

Part II

In this part we will investigate the continuity up to the boundary of Ω of the equation

$$\begin{cases} \beta'(u)u_t - \Delta u = 0, & u\big|_{\partial\Omega} = 0 \\ u(x,0) = u_0(x) \end{cases}$$

where $u_0 \in L^\infty(\Omega)$ and $\beta(s)$ is as before.

The only condition we imposed on the boundary of Ω is the "exterior sphere condition" which will be defined later. DiBenedetto in [7] has mentioned that the continuity up to the boundary for the plasma equation can be proved in a similar way as in his proof of the porous media equation. The proof we give here is however a much simpler proof. Indeed, we will show that the solution u goes to zero at the boundary at a "Lipschitz rate".

Definition: the boundary of Ω, $\partial\Omega$ is said to satisfy the exterior sphere condition if there exists $R > 0$, such that for each point $x_0 \in \partial\Omega$, there exists a ball $B_R(x^*)$ of radius R, with

$$B_R(x^*) \cap \bar{\Omega} = \{x_0\}$$

the geometric picture is as below.

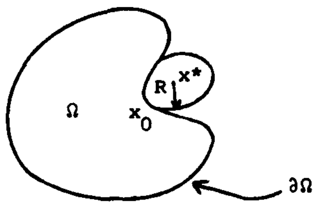


Figure II

<u>Theorem V-2</u>: Let u be the weak solution of

$$\begin{cases} \beta'(u)u_t - \Delta u = 0, & u\big|_{\partial\Omega} = 0 \\ u(x,0) = u_0(x) \end{cases}$$

where $u_0 \in L^\infty(\Omega)$ and there exists $M > m > 0$, such that $ms^{q-2} \le \beta'(s) \le Ms^{q-1}$. Furthermore, $\partial\Omega$ satisfies the exterior sphere condition with radius R, let $(x_0,T_0) \in \partial\Omega \times (0,\infty)$ and $(x,T) \in \Omega \times (0,T)$, then we have the following estimate:

$$u(x,T) \le \frac{Mk\,\|u_0\|_{L^\infty} 2^{N+1} C^{2N-1} d}{TR^{4N}}, \quad d = |x - x_0|$$

where k depends on M, $\|u_0\|_{L^\infty}$, N, R and Ω, and $R, |x| < C$ for all $x \in \Omega$.

<u>Proof</u>:

The proof is based on the idea of sub and supersolution and the construction of a barrier.

<u>Proof</u>:

1° Again, we consider the regularizations $\{u_n^\varepsilon\}$ of

$$\begin{cases} \beta'(u_n^\varepsilon)u_{n_t}^\varepsilon - \Delta u_n^\varepsilon = 0, & u_n^\varepsilon\big|_{\partial\Omega} = \varepsilon \\ u_n^\varepsilon(x,0) = u_0^n(x) + \varepsilon \end{cases}$$

Let $U_n^\varepsilon = u_n^\varepsilon - \varepsilon$ and $\beta_\varepsilon(s) = \beta(s + \varepsilon)$, then U_n^ε satisfies;

$$\begin{cases} \beta'_\varepsilon(U_n^\varepsilon)U^\varepsilon_{n_t} - \Delta U_n^\varepsilon = 0, \quad U_n^\varepsilon\big|_{\partial\Omega} = 0 \\ U_n^\varepsilon(x,0) = u_0^n \end{cases}$$

Multiplying by $\rho \in C^2(\bar{Q}_T)$, $\rho\big|_{\partial\Omega} = 0$ and integrate on $\Omega \times [0,T]$,

$$\int_\Omega \int_0^T \beta'_\varepsilon(U_n^\varepsilon)U^\varepsilon_{n_t}\rho - \int_0^T \int_\Omega \rho\Delta U_n^\varepsilon = 0$$

implying,

$$\int_\Omega \beta_\varepsilon(U_n^\varepsilon(T))\rho(T) - \int_\Omega \beta_\varepsilon(u_o^n)\rho(0) - \int_{Q_T} U_n^\varepsilon\Delta\rho + \beta_\varepsilon(U_n^\varepsilon)\phi_t = 0 .$$

2° Consider now any $x_0 \in \partial\Omega$, and without the loss of generality, assuming the center of the exterior sphere with respect to x_0 be at $(0,\ldots,0)$, we construct the following comparison function;

$$f(x,t) = \|u_0\|_{L^\infty(\Omega)}\left(1 - e^{-\frac{W(x)}{t}}\right)$$

where

$$W(x,t) = k\left[\frac{1}{R^{2N}} - \frac{1}{|x|^{2N}}\right]$$

k is to be determined later. We have,

$$f_t = -\|u_0\|_{L^\infty(\Omega)} e^{-\frac{W(x)}{t}} \frac{W(x)}{t^2}$$

$$\beta'_\varepsilon(f)f_t = \frac{-\beta'(f+\varepsilon)\|u_0\|_{L^\infty(\Omega)} W(x)e^{-\frac{W(x)}{t}}}{t^2}$$

$$\beta'_\varepsilon(f)f_t > -M\left[\|u_0\|_{L^\infty(\Omega)}\left(1 - e^{-\frac{W(x)}{t}}\right) + \varepsilon\right]^{q-2}\|u_0\|_{L^\infty(\Omega)} W(x)e^{-\frac{W(x)}{t}}$$

$$> -M\left[\|u_0\|_{L^\infty(\Omega)}\left(1 - e^{-\frac{W(x)}{t}}\right) + \varepsilon\right]^{q-2}\|u_0\|_{L^\infty(\Omega)} ke^{-\frac{W(x)}{t}} / R^{2N}t^2$$

$$\nabla f = \|u_0\|_{L^\infty(\Omega)} e^{-\frac{W(x)}{t}} \frac{\nabla W}{t}$$

$$\Delta f = \nabla\cdot\nabla f = -\|u_0\|_{L^\infty(\Omega)} e^{-\frac{W(x)}{t}} \frac{|\nabla W|^2}{t^2} + \|u_0\|_{L^\infty(\Omega)} e^{-\frac{W(x)}{t}} \frac{\Delta W}{t}$$

where

$$|\nabla W(x)|^2 = \frac{4N^2k^2}{|x|^{4N+2}},$$

$$-\Delta W(x) = -\left[\frac{2N^2k}{|x|^{2N+2}} - \frac{2Nk(N+2)}{|x|^{2N+2}}\right] = \frac{2N(N+2)k}{|x|^{2N+2}} > 0$$

Hence,

$$-\Delta f > \frac{4N^2k^2\|u_0\|_{L^\infty(\Omega)} e^{-\frac{W(x)}{t}}}{|x|^{4N+2}t^2}$$

implying,

$$\beta_\varepsilon'(f)f_t - \Delta f > 4N^2k^2\|u_0\|_{L^\infty(\Omega)} e^{-\frac{W(k)}{t}} / |x|^{4N+2}t^2$$

$$- M\|u_0\|^k_{L^\infty(\Omega)} e^{-\frac{W(x)}{t}} \left[\|u_0\|_{L^\infty(\Omega)}\left(1 - e^{-\frac{W(x)}{t}}\right) + \varepsilon\right]^{q-2} / k^{2N}t^2$$

$$= \frac{k\|u_0\|_{L^\infty(\Omega)} e^{-\frac{W(x)}{t}}}{t^2}\left\{\frac{4N^2k}{|x|^{4N+2}} - \frac{M\left[\|u_0\|_{L^\infty(\Omega)}\left(1 - e^{-\frac{W(x)}{t}}\right) + \varepsilon\right]^{q-2}}{R^{2N}}\right\} > 0$$

provided we choose k to be sufficiently large. We can do so because Ω is bounded.

3° $f(x,t)$ is a supersolution. Indeed, let ρ be a test function on Ω_T as before, by straight forward integration;

$$\int_\Omega \int_0^T \beta_\varepsilon'(f)f_t\rho - \int_0^T \int_\Omega \rho\Delta f > 0$$

$$\int_\Omega \beta_\varepsilon(f(T))\rho(T) - \int_\Omega \beta_\varepsilon(f(0))\rho(0) - \int_{Q_T} \beta_\varepsilon(f)\rho_t - \rho\Delta f \geq 0$$

$$\int_\Omega \beta_\varepsilon(f(T))\rho(T) \geq \int_\Omega \beta_\varepsilon(\|u_0^n\|_{L^\infty(\Omega)})\rho(0) + \int_{Q_T} \beta_\varepsilon(f)\rho_t - f\Delta\rho$$

Similarly as what we have done in Section IV,

$$\int_\Omega (\beta(U_n^\varepsilon(T) + \varepsilon) - \beta(f(T) + \varepsilon))^+ \leq 0$$

Thus,

$$U_n^\varepsilon(T) \leq f(T)$$

$$u_n^\varepsilon(x,T) \leq f(x,T) + \varepsilon = \|u_0\|_{L^\infty(\Omega)}\left(1 - e^{-\frac{W(x)}{T}}\right) + \varepsilon$$

for every $(x,T) \in \Omega \times (0,\infty)$.

4° Now let $(x_0,T_0) \in \partial\Omega \times (0,\infty)$ and $d = |x - x_0|$,

$$u_n^\varepsilon(x,T) \leq \|u_0\|_{L^\infty(\Omega)}\left(1 - e^{-\frac{W(x)}{T}}\right) + \varepsilon$$

for ρ small,

$$\|u_0\|_{L^\infty(\Omega)}\left(1 - e^{-\frac{W(x)}{T}}\right) + \varepsilon \approx \|u_0\|_{L^\infty(\Omega)}\left[1 - \left(1 - \frac{W(x)}{T}\right)\right] + \varepsilon$$

$$= \frac{k\|u_0\|_{L^\infty(\Omega)}}{T}\left[\frac{1}{R^{2N}} - \frac{1}{|x|^{2N}}\right] + \varepsilon$$

$$= \frac{k\|u_0\|_{L^\infty(\Omega)}}{TR^{2N}|x|^{2N}}\left[|x|^{2N} - R^{2N}\right] + \varepsilon$$

But,

$$|x|^{2N} - R^{2N} = (|x| - R)(|x|^{2N-1} + |x|^{2N-2}R + \dots + |x|R^{2N-2} + R^{2N-1})$$

$$\le (R + d - R)(2N)C^{2N-1} = 2NC^{2N-1}d$$

where R and $|x|$, $x \in \Omega$ are both less than C. Hence, for instance, let $d < R/2$, we have;

$$u_n^\varepsilon(x,T) \le \frac{\|u_0\|_{L^\infty(\Omega)} 2^{2N}(2N)C^{2N-1}d}{TR^{4N}} + \varepsilon$$

alternatively,

$$0 \le u_n^\varepsilon(x,T) - \varepsilon \le \frac{Nk\|u_0\|_{L^\infty(\Omega)} 2^{2N+1}C^{2N-1}d}{TR^{4N}}$$

where k is a constant depending on M, $\|u_0\|_{L^\infty(\Omega)}$, n, R and Ω. (One can easily check this by going back to the process of choosing k in step 2°)

Finally recalling that u_n^ε converges uniformly on the compact subsets of Q_T to u, we are done.

<u>Remarks:</u>

1° As before, if $u_0 \in L^{q-1}(\Omega)$ for $N > 2$ and $2 < q < \frac{2N-2}{N-2}$, then we still have the same kind of results due to regularizing effect.

2° In the above proof the condition $ms^{q-2} \le \beta'(s)$ for $s > 0$ was never used and so the constants do not depend on m.

3° What we have obtained in the proof of the theorem is not simply a decay rate of $u(x,t)$ at the boundary $\partial\Omega$ but also an uniform decay rate on the regularizations $\{u_n^\varepsilon\}$ at the boundary.

4° In the proof, all the estimates really depend only on M not m where $ms^{q-2} \le \beta'(s) \le Ms^{q-2}$ for $s > 0$.

SECTION VI. THE PERTURBATION OF THE POWER LAW (THE MAIN THEOREM)

In the previous sections, we have been studying the asymptotic profile of the plasma equation as we approach the extinction time T^*. In this section, we perturb the nonlinearity of the equation by a small amount and study the corresponding asymptotic behaviour of the solution. More precisely, we consider the nonlinear degenerate equation

$$(4) \qquad \begin{cases} \beta'(u)u_t - \Delta u = 0, \quad u\big|_{\partial\Omega} = 0 \\ u(0) = u_0 \end{cases}$$

where u_0 is some non-negative bounded initial data (or simply in $L^1(\Omega)$ for a certain range of q) and β satisfies,

($\beta 1$) $\beta \in C^1[0,\infty)$

($\beta 2$) $\beta(0) = \beta'(0) = 0$

($\beta 3$) There exists $M > m > 0$ such that

$$ms^{q-1} < \beta'(s) < Ms^{q-1}$$

and

$$\beta'(s) \to (q-1)s^{q-2} \quad \text{as} \quad s \to 0$$

Let us remark that the properties of the plasma equation used before, including the existence of T^*, regularity of solutions, uniqueness and the regularizing effect also hold for (4) (for the regularizing effect of the equation, see remark (iii) following Theorem IV-2). The proofs which work for the plasma equation can be carried over without any difficulties. The only thing which is unclear is whether there is an asymptotic profile or not. Since $\beta(s) \to s^{q-1}$ and $\beta'(s) \to (q-1)s^{q-2}$ as $s \to 0$, the nonlinearity behaves like that of the plasma equation as we approach the extinction time T^*. If $\beta'(s) \to (q-1)s^{q-2}$ (and hence $\beta(s) \to s^{q-1}$) sufficiently fast, we expect these take an asymptotic profile. Furthermore, intuitively speaking, the faster the solution decays to zero at T^*, the more (4) behaves like the plasma equation as $t \to T^*$. Hence, the decay rate of the solution is relevant for the analysis of the problem. Indeed, we do have an estimate of the decay rate of $\|u(t)\|_{L^\infty(\Omega)}$ as $t \to T^*$ which is provided by the modulus of

continuity of u, so before we proceed further, let us state some results concerning the modulus of continuity of u.

Remark: The interior regularity of porous media equation and that of the plasma equation have both been investigated by Paul Sacks in [31]. DiBenedetto in [7] investigated both the interior and boundary continuities of the porous media type equation. He also mentioned that the proofs in [7] can be modified to fit the case of the plasma equation (see the "significance and explanation" and Section V).

Theorem VI-1: Let Q be any subdomain of Q_T, for every $\varepsilon > 0$, there exists $\delta > 0$ depending on the data: N, $\max\{\|u_0\|_{L^\infty(\Omega)}, M\|u_0\|^{q-2}_{L^\infty(\Omega)}\}$, ms^{q-2}, ε, Ms^{q-2} and $\mathrm{dist}(Q,\partial Q_T)$ (here ∂Q_T being the boundary of the parabolic cylinder) such that for every (x_1,t_1), $(x_2,t_2) \in Q_T$, we have $|u(x_1,t_1) - u(x_2,t_2)| \le \varepsilon$ whenever $|(x_1,t_1) - (x_2,t_2)| \le \delta$.

This follows immediately from Theorem 1.1 of [31].

Theorem VI-2: Let the boundary of Ω satisfy the exterior sphere condition with radius R. Consider $\bar{\Omega} \times [t_0,\infty)$, $t_0 > 0$, there exists a continuous strictly increasing function $f^*|[0,\infty) \to [0,\infty)$ with $f^*(0) = 0$ and f^* depends on the data: N, ms^{q-2}, Ms^{q-2}, $\max\{\|u_0\|_{L^\infty(\Omega)}, M\|u_0\|^{q-2}_{L^\infty(\Omega)}\}$, diameter of Ω and t_0 such that

$$|u(x_1,t_1) - u(x_2,t_2)| \le f^*(|(x_1,t_1) - (x_2,t_2)|)$$

for every $(x_1,t_1),(x_2,t_2) \in \bar{\Omega} \times [t_0,\infty)$.

Proof:

It is sufficient to show that there exists $M_k \downarrow 0$, $R_k \downarrow 0$ depending on the data mentioned such that for every $(x_1,t_1),(x_2,t_2) \in \bar{\Omega} \times [t_0,\infty)$, we have,

$$|u(x_1,t_1) - u(x_2,t_2)| \le M_k$$

whenever

$$|(x_1,t_1) - (x_2,t_2)| \le R_k .$$

But this is true since we can always chop up the region $\bar{\Omega} \times [t_0,\infty)$ into an interior region and a boundary region. For the interior one, we invoke Theorem VI-1 while for the boundary one, we can handle it by Theorem V-2 of Section V.

Next, let us introduce the following class of nonlinearities:

$$\mathcal{B} = \{\beta | \beta \text{ satisfies } \beta(1), \beta(2) \text{ and } \beta(3)\} .$$

We are going to show that for a certain subset of $\mathcal{B}$, *if* $\beta'(s) \to (q-1)s^{q-2}$ as $s \to 0$ rapidly enough, then the results concerning the existence of an asymptotic profile remain true. More precisely:

<u>Theorem VI-3</u>: Let f^* be as in Theorem VI-2 and let $2 < q$ if $N \le 2$ or $2 < q < 2N/(N-2)$ if $N > 2$. Furthermore, suppose that:

(i) u_0 is non-negative and belong to $L^\infty(\Omega)$,

(ii) $\beta \in \mathcal{B}$,

(iii) $\limsup\limits_{s \to 0} \left|\dfrac{\beta'(s)}{(q-1)s^{q-2}} - 1\right| / [f^{*-1}(s)]^p < 1$ for some $p > 2/(q-2)$.

Then there exists an increasing sequence of times $t_n \to T^*$ and $p(\cdot,t_n) \to S(\cdot)$ in $\overset{\circ}{W}{}^1_2(\Omega)$ where $p(x,t) = u(x,t)/[(q-2)(T^*-t)]^{\frac{q}{q-2}}$ and S is a positive classical solution of (3).

<u>Remark</u>: It is trivial to see that $\beta(s) = s^{q-1}$ is not the only element of $\mathcal{B}$ which satisfies the hypothese of Theorem VI-3.

In this section, we will again invoke the nonlinear semi-group theory and a regularization of (4), so let us recall some basic results in these aspects.

<u>Lemma VI-1</u>: Let $\{u_n^\varepsilon\}$ be the solution of (4):

$$\begin{cases} \beta'(u_n^\varepsilon)u^\varepsilon_{n_t} - \Delta u_n^\varepsilon = 0, \quad u_n^\varepsilon\big|_{\partial\Omega} = \varepsilon \\ \\ u_n^\varepsilon(0) = u_0^n + \varepsilon \end{cases}$$

where $\varepsilon > 0$, $u_0^n \in C_0^\infty(\Omega)$, $u_0^n \to u_0$ in $L^p(\Omega)$ for every $1 \le p < \infty$ and $\|u_0^n\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}$. Then we have the estimate (which is due to Evans [23])

$$\|\Delta u_n^\varepsilon(t)\|_{L^1(\Omega)} \le \frac{C}{t^{(N+6)/4}} \|\beta(u_0^n)\|_{L^1(\Omega)}$$

where C depends on Ω, N and $M\|u_0\|^{q-2}_{L^\infty(\Omega)}$.

Finally by the Main theorem of [31] and Theorem V-2 of Section V, we have,

<u>Lemma VI-2</u>: The regularization $\{u_n^\varepsilon\}$ in Lemma VI-1 converge uniformly to $u(x,t)$ as $\varepsilon \to 0$ and $n \to \infty$ on $\bar{\Omega} \times [t_1,t_2]$ for any $0 < t_1 < t_2$.

<u>Proof</u>:

First we note that u_n^ε converge uniformly at the interior due to the Main theorem of [31] while from Theorem V-2 of Section V, we have an uniform decay rate of u_n^ε at the boundary of Ω. Hence, the convergence is uniform on $\bar{\Omega} \times [t_1,t_2]$.

Finally, by the continuous dependence property in the semi-group theory and the uniqueness of the weak solution, we can conclude that the limit of convergence is $u(x,t)$ where $v(x,t) = u(x,t)^{q-1}$ is the corresponding semi-group solution of (4).

Equipped with the above information, we are ready to prove the three key estimates as before.

The first key estimate is simply Proposition I-1 of Section I which we call it Lemma I' in this section.

<u>Lemma I'</u>: Let $2 < q$ if $N \leq 2$ or $2 < q < 2N/(N - 2)$ if $n > 2$, then there exists $C = C(q,\Omega)$ such that if u is the solution of (4) and T^* is its extinction time, we have,

$$(T^* - t)^{\frac{1}{q-2}} C(q,\Omega) \leq \left(\int_\Omega u(t)^q\right)^{1/q}$$

<u>Idea of Proof</u>:

Here one considers the function:

$$g(\alpha) = \int_0^\alpha s\beta'(s)ds ,$$

and gets an estimate on $\left(\int_\Omega g(u(t))dx\right)^{\frac{q-2}{q}}$ as in the proof of Lemma I for the plasma equation.

In the case of the other two key estimates, it turns out that they cannot be generalized from the estimates for the plasma equation as easily as above, and so we have to assume the conditions of Theorem VI-3. Also according to Evan's result in [23], we can

assume $u_0 \in L^\infty(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ as in the case of the plasma equation. (We can therefore also assume u_0^n of the regularization converge to u_0 in $\overset{\circ}{W}{}_2^1(\Omega)$.)

<u>Lemma II'</u>: Let u be the solution of (4) with the conditions of Theorem VI-3 being satisfied and $u_0 \in L^\infty(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$, then for every $s \in (0,T^*)$, we have,

$$\left(\int_\Omega u(t)^q dx\right)^{\frac{q-2}{q}} \le C(s,q,u_0,\Omega)(T^* - t)$$

for every $s \le t < T^*$.

<u>Proof</u>:

1° Let u_n^ε be the solution of the regularization

$$\beta'(u_n^\varepsilon)u_{n_t}^\varepsilon - \Delta u_n^\varepsilon = 0, \quad u_n^\varepsilon\big|_{\partial\Omega} = \varepsilon$$

$$u_n^\varepsilon(0) = u_0^n + \varepsilon$$

where $u_0^n \to u_0$ in $L^p(\Omega)$ and in $\overset{\circ}{W}{}_2^1(\Omega)$, $1 \le p < \infty$

$$\frac{d}{dt}\frac{\int_\Omega |\nabla u_n^\varepsilon|^2}{\left(\int_\Omega u_n^{\varepsilon^q}\right)^{2/q}} = \frac{2}{\left(\int_\Omega u_n^{\varepsilon^q}\right)^{\frac{q+2}{q}}}\left[\left(\int_\Omega u_n^{\varepsilon^q}\right)\left(\int_\Omega \nabla u_n^\varepsilon \cdot \nabla u_{n_t}^\varepsilon\right) - \left(\int_\Omega |\nabla u_n^\varepsilon|^2\right)\left(\int_\Omega u_n^{\varepsilon^{q-1}} u_{n_t}^\varepsilon\right)\right]$$

$$= \frac{2}{\left(\int_\Omega u_n^{\varepsilon^q}\right)^{\frac{q+2}{q}}}\left[\left(-\int_\Omega u_n^{\varepsilon^q}\right)\left(\int_\Omega u_{n_t}^\varepsilon \Delta u_n^\varepsilon\right) + \left(\int_\Omega u_n^{\varepsilon^{q-1}} u_{n_t}^\varepsilon\right)\left(\int_\Omega u_n^\varepsilon \Delta u_n^\varepsilon - \varepsilon\int_\Omega \Delta u_n^\varepsilon\right)\right]$$

$$= \frac{2}{\left(\int_\Omega u_n^{\varepsilon^q}\right)^{\frac{q+2}{q}}}\left[\left(-\int_\Omega u_n^{\varepsilon^q}\right)\left(\int_\Omega u_{n_t}^{\varepsilon^2}\beta'(u_n^\varepsilon)\right) + \left(\int_\Omega u_n^{\varepsilon^{q-1}} u_{n_t}^\varepsilon\right)\left(\int_\Omega (q-1)u_n^{\varepsilon^{q-1}} u_{n_t}^\varepsilon\right) + II + III\right]$$

$$\le \frac{2}{\left(\int_\Omega u_n^\varepsilon(t)^q\right)^{\frac{q+2}{q}}}[I + II + III]$$

by Hölder's inequality where,

$$I = \left(\int_\Omega u_n^\varepsilon(t)^q\right)\left[\int_\Omega \left((q-1)u_n^{\varepsilon^{q-2}} - \beta'(u_n^\varepsilon)\right)u_{n_t}^{\varepsilon^2}\right]$$

$$II = \left(\int_\Omega u_n^{\varepsilon^{q-1}} u_{n_t}^\varepsilon\right)\left(\int_\Omega u_n^\varepsilon\left(\beta'(u_n^\varepsilon) - (q-1)u_n^{\varepsilon^{q-2}}\right)u_{n_t}^\varepsilon\right)$$

$$III = \left(-\varepsilon\int_\Omega \Delta u_n^\varepsilon(t)\right)\left(\int_\Omega u_n^{\varepsilon^{q-1}} u_{n_t}^\varepsilon\right)$$

Let us now estimate them separately.

2°

$$|I|/\left(\int_\Omega u_n^\varepsilon(t)^q\right)^{\frac{q+2}{q}} = \int_\Omega \left|\frac{(q-1)u_n^{\varepsilon^{q-2}} - \beta'(u_n^\varepsilon)}{\beta'(u_n^\varepsilon)}\right|\beta'(u_n^\varepsilon)u_{n_t}^{\varepsilon^2} \Big/ \left(\int_\Omega u_n^\varepsilon(t)^q\right)^{2/q}$$

$$< \left\|\frac{(q-1)u_n^{\varepsilon^{q-2}} - \beta'(u_n^\varepsilon)}{\beta'(u_n^\varepsilon)}\right\|_{L^\infty(\Omega)} \int_\Omega \beta'(u_n^\varepsilon)u_{n_t}^{\varepsilon^2} \Big/ \left(\int_\Omega u_n^{\varepsilon^q}(t)\right)^{2/q}$$

$$< \|f^{*^{-1}}(u_n^\varepsilon(t))\|^p_{L^\infty(\Omega)}\left(\int_\Omega \beta'(u_n^\varepsilon)u_{n_t}^{\varepsilon^2}\right) \Big/ \left(\int_\Omega u_n^{\varepsilon^q}(t)\right)^{2/q}$$

for $0 < s < t < t' < T^*$.

$$|I| \Big/ \left(\int_\Omega u_n^\varepsilon(t)^q\right)^{\frac{q+2}{q}}$$

$$<\left(\|f^{*^{-1}}(u)\|^p_{L^\infty(\Omega)} \Big/ \left(\int_\Omega u(t)^q\right)^{2/q} + \delta(\varepsilon,n,s,t')\right)\left(\int_\Omega \beta'(u_n^\varepsilon)u_{n_t}^{\varepsilon^2}\right)$$

where $\delta(\varepsilon,n,s,t') \to 0$ as $\varepsilon \to 0$ and $n \to \infty$ due to Lemma 6.2. But we know that $u(x,T^*) = 0$ for every $x \in \bar{\Omega}$ and in view of Lemma I', we have,

$$\|f^{*-1}(u(t))\|^{p}_{L^\infty(\Omega)} \Big/ \left(\int_\Omega u(t)^q\right)^{2/q} \le [f^{*-1}(f^*(T^* - t))]^p / \, C(T^* - t)^{\frac{2}{q-2}}$$

$$\le C(T^* - t)^p / \, (T^* - t)^{\frac{2}{q-2}} \le C \, .$$

where $C = C(M,m,\Omega,q)$. Hence,

$$|I| \Big/ \left(\int_\Omega u_n^\varepsilon(t)^q\right)^{\frac{q+2}{q}} \le (C + \delta(\varepsilon,n,s,t')) \int_\Omega \beta'(u_n^\varepsilon) {u_{n_t}^\varepsilon}^2$$

3° For II, similarly as in 2°,

$$|II| \Big/ \left(\int_\Omega u_n^\varepsilon(t)^q\right)^{\frac{q+2}{q}} \le \frac{\int_\Omega |u_n^\varepsilon|^{q-1} u_{n_t}^\varepsilon \, \|f^{*-1}(u_n^\varepsilon(t))\|^p_{L^\infty(\Omega)} \int_\Omega |u_n^\varepsilon \beta'(u_n^\varepsilon) u_{n_t}^\varepsilon|}{\left(\int_\Omega u_n^\varepsilon(t)^q\right)^{\frac{q+2}{q}}}$$

But

$$\int_\Omega |u_n^\varepsilon \beta'(u_n^\varepsilon) u_{n_t}^\varepsilon| \le \sqrt{\int_\Omega \beta'(u_n^\varepsilon) {u_{n_t}^\varepsilon}^2} \sqrt{\int_\Omega u_n^\varepsilon(t)^2 \beta'(u_n^\varepsilon(t))}$$

thus,

$$|II| \Big/ \left(\int_\Omega u_n^\varepsilon(t)^q\right)^{\frac{q+2}{q}}$$

$$\le C(M,m,q) \|f^{*-1}(u_n^\varepsilon(t))\|^p_{L^\infty(\Omega)} \int_\Omega \beta'(u_n^\varepsilon) {u_{n_t}^\varepsilon}^2 \Big/ \left(\int_\Omega u_n^\varepsilon(t)^q\right)^{2/q}$$

$$\le C(M,m,q)(C + \delta(\varepsilon,m,s,t')) \int_\Omega \beta'(u_n^\varepsilon) {u_{n_t}^\varepsilon}^2$$

where $C = C(M,m,q,\Omega)$ as before.

4° For III, we have,

$$|III| \Big/ \left(\int_\Omega u_n^{\varepsilon^q}(t)\right)^{\frac{q+2}{q}} \le \varepsilon\left(\int_\Omega |\Delta u_n^\varepsilon(t)|\right)\left(\int_\Omega |u_n^{\varepsilon^{q-1}} u_{n_t}^\varepsilon|\right) \Big/ \left(\int_\Omega u_n^\varepsilon(t)^q\right)^{\frac{q+2}{q}}$$

$$\le \frac{\varepsilon\left(\int_\Omega |\Delta u_n^\varepsilon(t)|\right)\sqrt{\int_\Omega |u_n^{\varepsilon^2}(t)\beta'(u_n^\varepsilon(t))| \int_\Omega \beta'(u_n^\varepsilon)u_{n_t}^{\varepsilon^2}}}{\left(\int_\Omega u_n^{\varepsilon^q}(t)\right)^{\frac{q+2}{q}}}$$

$$\le \frac{\varepsilon C(M,m)\sqrt{\int_\Omega \beta'(u_n^\varepsilon)u_{n_t}^{\varepsilon^2}} \int_\Omega |\Delta u_n^\varepsilon(t)|}{\left(\int_\Omega u_n^\varepsilon(t)^q\right)^{1/2+2/q}}$$

But by Lemma 6.1

$$\|\Delta u_n^\varepsilon(t)\|_{L^1(\Omega)} \le \frac{C(M,N,\Omega,u_0)}{t^{(N+6)/4}} \|\beta(u_0^n)\|_{L^1(\Omega)}$$

and so,

$$|III| \Big/ \left(\int_\Omega u_n^\varepsilon(t)^q\right)^{\frac{q+2}{q}} \le \frac{\varepsilon C(M,m,\Omega,u_0)\sqrt{\int_\Omega \beta'(u_n^\varepsilon)u_{n_t}^{\varepsilon^2}}}{s^{(N+6)/4}\left((T^* - t')^{\frac{q+2}{2}} - \delta(\varepsilon,n,s,t')\right)}$$

for every $t \in [s,t']$ and $\delta(\varepsilon,n,s,t') \to 0$ as $\varepsilon \to 0$, $n \to \infty$. Integrating, we have for $s \le t \le t'$,

$$\frac{\int_\Omega |\nabla u_n^\varepsilon(t)|^2}{\left(\int_\Omega u_n^\varepsilon(t)^q\right)^{2/q}} - \frac{\int_\Omega |\nabla u_n^\varepsilon(s)|^2}{\left(\int_\Omega u_n^\varepsilon(s)^q\right)^{2/q}} \le 2(C(M,m,q,\Omega) + \delta(\varepsilon,n,s,t'))\int_s^t \int_\Omega \beta'(u_n^\varepsilon)u_{n_t}^{\varepsilon^2}$$

$$+ \frac{\varepsilon C(M,m,q,u_0)\sqrt{\int_s^t \int_\Omega \beta'(u_n^\varepsilon)u_{n_t}^{\varepsilon^2}}}{s^{(N+6)/4}\left((T^* - t')^{\frac{q+2}{2}} - \delta(\varepsilon,n,s,t')\right)}$$

Furthermore observing that,

$$\int_s^t \int_\Omega \beta'(u_n^\varepsilon) u_{n_t}^{\varepsilon^2} = \int_s^t \int_\Omega u_{n_t}^\varepsilon \Delta u_n^\varepsilon = \int_s^t \int_\Omega (u_n^\varepsilon - \varepsilon)_t \Delta(u_n^\varepsilon - \varepsilon)$$

$$= -\int_s^t \frac{d}{dt}\int_\Omega |\nabla u_n^\varepsilon|^2 = \frac{1}{2}\int_\Omega |\nabla u_n^\varepsilon(s)|^2 - \frac{1}{2}\int_\Omega |\nabla u_n^\varepsilon(t)|^2$$

$$< \frac{1}{2}\int_\Omega |\nabla u_n^\varepsilon(s)|^2 < \frac{1}{2}\int_\Omega |\nabla u_0^n|^2$$

Hence,

$$\int_\Omega |\nabla u_n^\varepsilon(t)|^2 \Big/ \Big(\int_\Omega u_n^{\varepsilon^q}(t)\Big)^{2/q} < 2(C(N,m,\Omega,q) + \delta(\varepsilon,n,s,t')) \int_\Omega |\nabla u_0^n|^2$$

$$+ \int_\Omega |\nabla u_0^n|^2 \Big/ \Big(\int_\Omega u_n^\varepsilon(s)^q\Big)^{2/q} + \frac{\varepsilon C(N,m,q,\Omega,u_0)\sqrt{\int_\Omega |\nabla u_0^n|^2}}{s^{(N+6)/4}\left((T^* - t')^{\frac{q+2}{2}} - \delta(\varepsilon,n,s,t')\right)}$$

for every $t \in [s,t']$ where $0 < s < t' < T^*$.

6° Finally, as in the proof of $\beta(s) = s^{q-1}$ case,

$$-\frac{d}{dt}\Big(\int_\Omega (u_n^\varepsilon(t) - \varepsilon)^q\Big)^{\frac{q-2}{q}} < (q-1)\,\frac{\left(\int_\Omega |\nabla u_n^\varepsilon(t)|^2\right)}{\left(\int_\Omega u_n^\varepsilon(t)^q\right)^{2/q}} \cdot \frac{\left(\int_\Omega u_n^\varepsilon(t)^q\right)^{2/q}}{\left(\int_\Omega (u_n^\varepsilon(t) - \varepsilon)^q\right)^{2/q}}$$

Integrating,

$$\Big(\int_\Omega (u_n^\varepsilon(t) - \varepsilon)^q\Big)^{\frac{q-2}{q}} - \Big(\int_\Omega (u_n^\varepsilon(t') - \varepsilon)^q\Big)^{\frac{q-2}{q}}$$

$$< (q-1)(t' - t)(1 + \delta(\varepsilon,n))\Big[2(C(N,m,q,\Omega) + \delta(\varepsilon,n,s,t'))\int_\Omega |\nabla u_0^n|^2$$

$$+ \int_\Omega |\nabla u_0^n|^2 \Big/ \Big(\int_\Omega u_n^\varepsilon(s)^q\Big)^{2/q} + \frac{\varepsilon C(N,m,q,\Omega)\sqrt{\int_\Omega |\nabla u_0^n|^2}}{s^{(N+6)/4}\left((T^* - t)^{\frac{q}{2}+2} - \delta(\varepsilon,n,s,t')\right)}\Big]$$

On letting $\varepsilon \to 0$ and $n \to \infty$

$$\left(\int_\Omega u(t)^q\right)^{\frac{q-2}{q}} - \left(\int_\Omega u(t')^q\right)^{\frac{q-2}{q}} \le (q-1)\left[C(M,m,q,\Omega)\int_\Omega |\nabla u_0|^2 + \frac{\int_\Omega |\nabla u_0|^2}{\left(\int_\Omega u(s)^q\right)^{2/q}}\right](t'-t)$$

$$\le C(M,m,q,\Omega)\left[\int_\Omega |\nabla u_0|^2\left(1 + \frac{1}{(T^*-s)^{\frac{2}{q-2}}}\right)\right](t'-t)$$

Now, let $t' \to T^*$, we have,

$$\left(\int_\Omega u(t)^q\right)^{\frac{q-2}{q}} \le C(M,m,q,\Omega)\left[\left(\int_\Omega |\nabla u_0|^2\right)\left(1 + \frac{1}{(T^*-s)^{\frac{2}{q-2}}}\right)\right](T^*-t)$$

for every $t > s$, $0 < s < T^*$.

Lemma III': Let u be the solution of (4) with the hypothese of Theorem 3.6 being satisfied, and also let

$$W(x,T) = u(x,t)/((T^*-t)(q-2))^{\frac{1}{q-2}}, \quad t = T^*(1 - e^{-(q-2)T})$$

then,

(i) $W_T = \dfrac{(\Delta W)(T^* e^{-(q-2)T})}{\beta'(u)} + W$

i.e.

$$\left[\frac{\beta'(u)}{(q-1)u^{q-2}}\right](q-1)W^{q-2}W_T = \Delta W + \left[\frac{\beta'(u)}{(q-1)u^{q-2}}\right](q-1)W^{q-1}$$

and

(ii) there exists $\{T_n\}$ such that, as $T_n \to \infty$,

$$\|W^{q-2}W_T(T_n)\|_{L^{\frac{q}{q-1}}(\Omega)} \to 0$$

$\left(\text{Alternatively, } \left\|\dfrac{\beta'(u)}{(q-2)T^* e^{-(q-2)T_n}} W_T(T_n)\right\|_{L^{\frac{q}{q-1}}(\Omega)} \to 0\right)$

Proof:

1° (i) is immediately obvious by straight forward computations. To prove (ii), let $\{u_n^\varepsilon\}$ be as before. We define,

$$W_n^\varepsilon = u_n^\varepsilon/((T^* - t)(q-2))^{\frac{1}{q-2}} = u_n^\varepsilon/(T^*(q-2))^{\frac{1}{q-2}}e^{-T}$$

W_n^ε satisfy exactly the same equation as in (i) and so,

$$\frac{\beta'(u_n^\varepsilon)W_{n_T}^\varepsilon}{(T^*(q-2))e^{-(q-2)T}} = \Delta W_n^\varepsilon + \frac{\beta'(u_n^\varepsilon)W_n^\varepsilon}{(T^*(q-2))e^{-(q-2)T}} .$$

Multiply by $W_{n_T}^\varepsilon$ and integrating,

$$\int_\Omega \frac{\beta'(u_n^\varepsilon){W_{n_T}^\varepsilon}^2}{(T^*(q-2))e^{-(q-2)T}} < \int_\Omega W_{n_T}^\varepsilon \Delta W_n^\varepsilon + \int_\Omega \frac{|\beta'(u_n^\varepsilon) - (q-1){u_n^\varepsilon}^{q-2}|W_n^\varepsilon W_{n_T}^\varepsilon}{(T^*(q-2))e^{-(q-2)T}}$$

Let us denote

$$\int_\Omega \frac{|\beta'(u_n^\varepsilon) - (q-1){u_n^\varepsilon}^{q-2}||W_n^\varepsilon W_{n_T}^\varepsilon|}{(T^*(q-2))e^{-(q-2)T}}$$

by I_n^ε and estimate it.

2° Recalling that $t = T^*(1 - e^{-(q-2)T})$, $(T^* - t)^{\frac{1}{q-2}} = (T^*)^{\frac{1}{q-2}}e^{-T}$,

$$W_{n_T}^\varepsilon = \frac{u_{n_t}^\varepsilon(x,t)(T^*(q-2))e^{-(q-2)T}}{\left(T^*(q-2)e^{-(q-2)T}\right)^{\frac{1}{q-2}}} + \frac{u_n^\varepsilon(x,t)}{\left(T^*(q-2)e^{-(q-2)T}\right)^{\frac{1}{q-2}}}$$

we have,

$$I_n^\varepsilon < I_1 + I_2$$

$$I_1 = \int_\Omega \frac{|\beta'(u_n^\varepsilon) - (q-1){u_n^\varepsilon}^{q-2}||u_n^\varepsilon(t)||u_{n_t}^\varepsilon(t)|}{\left(T^*(q-2)e^{-(q-2)T}\right)^{\frac{2}{q-2}}}$$

$$I_2 = \int_\Omega \frac{|\beta'(u_n^\varepsilon) - (q-1){u_n^\varepsilon}^{q-2}|W_n^\varepsilon(t)^2}{(T^*(q-2))e^{-(q-2)T}}$$

For I_1,

$$I_1 < C(M,m)\|u_n^\varepsilon(f^{*-1}(u_n^\varepsilon))^p\|_{L^\infty(\Omega)} \int_\Omega |\beta'(u_n^\varepsilon(t))u_{n_t}^\varepsilon(t)|/(T^* - t)^{\frac{2}{q-2}}$$

$$= C(M,m)\|u_n^\varepsilon(f^{*-1}(u_n^\varepsilon))^p\|_{L^\infty(\Omega)} \int_\Omega |\Delta u_n^\varepsilon(t)|/(T^* - t)^{\frac{2}{q-2}}$$

Let $0 < t_1 < t < t_2 < T^*$

$$\|u_n^\varepsilon(t)(f^{*-1}(u_n^\varepsilon(t)))^p\|_{L^\infty(\Omega)} \to \|u(t)(f^{*-1}(u(t)))^p\|_{L^\infty(\Omega)}$$

uniformly on $[T_1,t_2]$ due to Lemma 6.2, together with Lemma 6.1, we have,

$$I_1 < \frac{C(M,m,N,\Omega,u_0)}{(T^* - t)^{\frac{2}{q-2}}} \left(\|u(t)(f^{*-1}(u(t)))^p\|_{L^\infty(\Omega)} + \delta(\varepsilon,n,t_1,t_2)\right) \frac{\|\beta(u_0^n)\|_{L^1(\Omega)}}{t^{(N+6)/4}}$$

where $\delta(\varepsilon,n,t_1,t_2) \to 0$ as $\varepsilon \to 0$ and $n \to \infty$.

3° In the same way,

$$I_2 < \int_\Omega \left|\frac{\beta'(u_n^\varepsilon) - (q - 1)u_n^{\varepsilon\, q-2}}{\beta'(u_n^\varepsilon)}\right| \frac{\beta'(u_n^\varepsilon)W_n^\varepsilon(t)^2}{T^*(q - 2)e^{-(q-2)T}}$$

$$< C(M,m)\|(f^{*-1}(u_n^\varepsilon(t)))^p\|_{L^\infty(\Omega)} \int_\Omega W_n^\varepsilon(t)^q$$

$$< C(M,m)\left(\|(f^{*-1}(u(t)))^p\|_{L^\infty(\Omega)} + \delta(\varepsilon,n,t_1,t_2)\right) \int_\Omega W_n^\varepsilon(t)^q$$

But $\int_\Omega W(t)^q$ is uniformly bounded by say C' where $C' = C'(M,m,q,\Omega,u_0,T^*)$ due to Lemma II' and $\int_\Omega W_n^\varepsilon(t)^q \to \int_\Omega W(t)^q$ uniformly on $[t_1,t_2]$. Hence,

$$I_2 < C(M,m)\left(\|(f^{*-1}(u(t)))^p\|_{L^\infty(\Omega)} + \delta(\varepsilon,n,t_1,t_2)\right)(C' + \delta'(\varepsilon,n,t_1,t_2))$$

where both δ and $\delta' \to 0$ as $\varepsilon \to 0$ and $n \to \infty$. Now that on $[t_1,t_2]$

$$I_n^\varepsilon \le \frac{C(M,m,N,\Omega,u_0)}{(T^* - t)^{\frac{2}{q-2}}}\left(\|u(t)(f^{*-1}(u(t)))^p\|_{L^\infty(\Omega)} + \delta(\varepsilon,n,t_1,t_2)\right)\frac{\|\beta(u_0^n)\|_{L^1(\Omega)}}{t_1^{(N+6)/4}}$$

$$+ C(M,m)\left(\|(f^{*-1}(u(t)))^p\|_{L^\infty(\Omega)} + \delta(\varepsilon,n,t_1,t_2)\right)(C' + \delta'(\varepsilon,n,t_1,t_2))$$

4° By Hölder's inequality

$$\int_\Omega |W_n^{\varepsilon\,q-2}(T)W_{n_T}^\varepsilon(T)|^{\frac{q}{q-1}} \le \left(\int_\Omega W_n^\varepsilon(T)^q\right)^{\frac{q-2}{2(q-1)}}\left[\int_\Omega W_n^{\varepsilon\,q-2}(T)W_{n_T}^\varepsilon(T)^2\right]^{\frac{q}{2(q-1)}}$$

$$\le (C' + \delta'(\varepsilon,n,t_1,t_2))^{\frac{q-2}{2(q-1)}}\left[\int_\Omega \frac{\beta'(u_n^\varepsilon)W_{n_T}^{\varepsilon\,2}}{T^*(q-2)e^{-(q-2)T}}\right]^{\frac{q}{2(q-1)}}$$

Thus,

$$\|W_n^{\varepsilon\,q-2}W_{n_T}^\varepsilon(T)\|^2_{L^{\frac{q}{q-1}}(\Omega)} \le (C' + \delta'(\varepsilon,n,t_1,t_2))^{\frac{q-2}{q}}\left[\int_\Omega \frac{\beta'(u_n^\varepsilon)W_{n_T}^{\varepsilon\,2}}{T^*(q-2)e^{-(q-2)T}}\right]$$

$$\le (C' + \delta'(\varepsilon,n,t_1,t_2))^{\frac{q-2}{q}}\left[\frac{d}{dT}\left(-\frac{1}{2}\int_\Omega |\nabla W_n^\varepsilon(T)|^2\right) + I_n^\varepsilon + \left(\frac{q-1}{q}\right)\frac{d}{dT}\left(\int_\Omega W_n^\varepsilon(T)^q\right)\right]$$

by step 1°.

5° Now let T_1 and T_2 be such that,

$$t_1 = T^*(1 - e^{-(q-2)T_1}), \quad t_2 = T^*(1 - e^{-(q-2)T_2})$$

then,

$$\int_{T_1}^{T_2} \| W_n^{\varepsilon^{q-2}} W_{n_T}^{\varepsilon}(T)\|^2_{L^{\frac{q}{q-1}}(\Omega)} \le (C' + \delta'(\varepsilon,n,t_1,t_2))^{\frac{q-2}{q}} \Big[\frac{1}{2}\int_\Omega |\nabla W_n^{\varepsilon}(T_1)|^2$$

$$- \frac{1}{2}\int_\Omega |\nabla W_n^{\varepsilon}(T_2)|^2 + \Big(\frac{q-1}{q}\Big)\int_\Omega W_n^{\varepsilon}(T_2)^q - \Big(\frac{q-1}{q}\Big)\int_\Omega W_n^{\varepsilon}(T_1)^q + \int_{T_1}^{T_2} I_n^{\varepsilon}(T)\Big]$$

We note that $\int_\Omega W_n^{\varepsilon}(T_2)^q \to \int_\Omega W(T_2)^q$ which is bounded by Lemma II', so that we only need to estimate $\int_\Omega |\nabla W_n^{\varepsilon}(T_1)^2|$ and $\int_{T_1}^{T_2} I_n^{\varepsilon}(T)$.

6° To estiamte $\int_\Omega |\nabla W_n^{\varepsilon}(T_1)|^2$, we note that by Lemma I', it is sufficient to estimate

$$\frac{\int_\Omega |\nabla u_n^{\varepsilon}(T_1)|^2}{(\int_\Omega u_n^{\varepsilon}(T_1)^q)^{2/q}}$$

But this is bounded as shown in the proof of Lemma II'. Hence $\int_\Omega |\nabla W_n^{\varepsilon}(t)|^2$ is uniformly bounded on $t > t_1$ and hence on $T > T_1$ as $n \to \infty$ and $\varepsilon \to$

7° For $\int_{T_1}^{T_2} I_n^{\varepsilon}(T)dT$, we have

$$\int_{T_1}^{T_2} I_n^{\varepsilon}(T)dT \le \int_{T_1}^{T_2} \frac{C(M,m,N,\Omega,u_0)\|\beta(u_0)\|_{L^1(\Omega)} e^{2T}}{t_1^{(N+6)/4}(q-2)^{\frac{2}{(q-2)}}} \left(\|u(f^{*-1}(u))^{p}\|_{L^\infty(\Omega)} + \delta(\varepsilon,n,t_1,t_2)\right)dT$$

$$+ \int_{T_1}^{T_2} C(M,m)\left(\|(f^{*-1}(u(T)))^{p}\|_{L^\infty(\Omega)} + \delta(\varepsilon,n,t_1,t_2)\right)(C' + \delta'(\varepsilon,n,t_1,t_2))dT$$

Let $n \to \infty$, $\varepsilon \to 0$, then $W_n^{\varepsilon^{q-2}} W_{n_T}^{\varepsilon} \dashrightarrow h$ for some h in $L^2(T_1,T_2|L^{\frac{q}{q-1}}(\Omega))$ and,

$$\int_{T_1}^{T_2} \|h(T)\|^2_{L^{\frac{q}{q-1}}(\Omega)} dT < (C')^{\frac{q-2}{q}} \Big[\frac{\|u_0\|_{L^\infty(\Omega)} \|\beta(u_0)\|_{L^1(\Omega)}}{t_1^{(N+6)/4}} + \left(\frac{q-1}{q}\right) \int_\Omega W(T_1)^q$$

$$+ \int_{T_1}^{T_2} C(M,m,N,\Omega,u_0)\|\beta(u_0)\|_{L^1(\Omega)} \|u(f^{*-1}(u))^p\|_{L^\infty(\Omega)} e^{2T} dT$$

$$+ C(M,m) \int_{T_1}^{T_2} C'\|(f^{*-1}(u))^p\|_{L^\infty(\Omega)} dT\Big] .$$

The only thing we still have to show is $h = W^{q-2}W_T$.

8° We can show $h = W^{q-2}W_T$ in exactly the same way as in the case of the plasma equation. Every step of the proof for the plasma equation can be carried over as far as $\int_{T_1}^{T_2} \int_\Omega |\nabla W_n^\varepsilon(T)|^2$ is bounded uniformly in n and ε. This in turn boils down to showing the boundedness of $\int_\Omega |\nabla W_n^\varepsilon(T)|^2$ for $T_1 < T < T_2$, but this is true as was mentioned in 6°. Thus, $h = W^{q-2}W_T$.

Finally, let us recall that $\|u(t)\|_{L^\infty(\Omega)} < f^*((T^* - t))$ where $T^* - t = (T^*)e^{-(q-2)T}$, we have

$$\|u(T)(f^{*-1}(u(T)))^p\|_{L^\infty(\Omega)} < \|u_0\|_{L^\infty(\Omega)} (T^*(q-2))^p e^{-(q-2)pT}$$

$$\|(f^{*-1}(u(T)))^p\|_{L^\infty(\Omega)} = (T^*(q-2))^p e^{-(q-2)pT}$$

and so

$$\int_{T_1}^{T_2} \|h(T)\|^2_{L^{\frac{q}{q-1}}(\Omega)} \, dT < (C')^{\frac{q-2}{q}} \Big[\frac{\|u_0\|_{L^\infty(\Omega)} \|\beta(u_0)\|_{L^1(\Omega)}}{t_1^{(N+6)/4}} + \Big(\frac{q-1}{q}\Big) \int_\Omega W(T_1)^q$$

$$+ (T^*(q-2))^p \int_{T_1}^{T_2} C(M,m,N,\Omega,u_0) \|\beta(u_0)\|_{L^1(\Omega)} e^{-p(q-2)T} dT$$

$$+ C'C(M,m) \int_{T_1}^{T_2} (T^*(q-2))^p e^{-p(q-2)T} dT\Big] .$$

Observing that the right hand side is uniformly bounded with respect to T_2, we can let $T_2 \to \infty$ to obtain the desired conclusion.

Equipped with Lemmas I', II' and III', Theorem VI-3 is now immediate as in the case of the plasma equation.

Remark:

(i) We have assumed the initial data to be bounded. However, all we need is the initial data to be in $L^1(\Omega)$ for $2 < q < (2N-2)/(N-2)$, $N > 2$ due to regularizing effect.

(ii) The decay rate of $\|u(t)\|_{L^\infty(\Omega)}$ as $t \to T^*$ is still unknown. If we could obtain an explicit estimate on this rate rather than just having a rough estimate provided by the modulus of continuity o´ u (which the author suspects is not sharp in this regard), then it is obvious that we can obtain a sharper and more explicit result.

Acknowledgement: I would like to thank Professor M. G. Crandall sincerely for his kind and helpful guidance in this research.

REFERENCES

1. Aronson, D. G., Regularity properties of flows through porous media, SIAM J. Applied Math., 17 (1969) 461-467.
2. Aronson, D. G., Regularity properties of flows through porous media. The interface, Arch. Rat. Mech. Anal., 37 (1970) 1-10.
3. Aronson, D. G., Regularity of flows through porous media. A counterexample SIAM J. Applied Math 19 (1970) 299-307.
4. Aronson, D. G., Crandall, M. G. and Peletier, L. A., Stabilization of a degenerate nonlinear diffusion problem, Math Research Center TSR 2220.
5. Aronson, D. G. and Peletier, L. A., Large time behaviour of solutions of the porous media equation in bounded domains, J. Diff. Eqn. 39 (1981) 378-412.
6. Barbu, V., Nonlinear semi-groups and differential equations in Banach spaces, Nordhoff International Publishing Co., Leyden (1976).
7. DiBenedetto, E., Continuity of weak solutions to a general porous medium equation, TSR #2189, Math Research Center, Madison, Wisconsin.
8. Crandall, M. G. and Benilan, P., The continuous dependence on ϕ of solutions of $u_t - \Delta\phi(u) = 0$, Indiana University Math. J., Vol. 30, No. 2 (1981).
9. Benilan, P. and Crandall, M. G., (to appear).
10. Crandall, M. G., Pazy, A. and Benilan, P., Nonlinear Evolution governed by accretive operators (to appear).
11. Benilan, P. and Veron, L., Coercivité et propriétés régularisantes des semi-groupes nonlinéaires dans les espacves de Banach, Publ. Math. de l'Univ. de Besaucon (1976).
12. Berryman, James G. and Holland, Charles J., Stability of the separable solution for fast diffusion, Archive for Rational Mechanic and Analysis, 1980.
13. Bertsch, M., Asymptotic behaviour of solutions of a nonlinear diffusion equation, SIAM J. Appl. Math. 42 (1982) 66-76.
14. Bertsch, M. and Peletier, L. A., The asymptotic profile of solutions of degenerate diffusion equations, to appear in Arch. Rat. Mech. Anal.

15. Brezis, H. and Crandall, M. G., Uniqueness of solutions of the initial value problem for $u_t - \Delta\phi(u) \ni 0$, J. Math pures et. appl. 58 (1979) 153-163.

16. Brezis, H. and Strauss, W., Semi-linear second order elliptic equations in L^1, J. Math. Soc., Japan 25 (1973).

17. Caffarelli, L. A. and Evans, C. L., Continuity of the temperature in the two-phase Stefan problem.

18. Caffarelli, L. A. and Friedman, A., Continuity of the density of a gass flow in a porour medium, Trans. Amer. Math. Soc. 252 (1979).

19. Caffarelli, L. A. and Friedman, A., Regularity of the free boundary of a gas flow in an n-dimensional porous medium, Indiana Univ. J. of Math. vol. 29 #3 (1980).

20. Chicco., Solvability of the Dirichlet problem in $H^{2,p}(\Omega)$ for a class of linear second order elliptic equations, Boll. UMI4 (1977), 374-387.

21. Crandall, M. G., An introduction to evolution governed by accretive operators in "Proceedings of the International Symposium on Dynamical Systems", Brown University, Providence, RI, Aug. 1974.

22. Diaz Diaz, G. and Diaz Diaz, J. I., Finite Extinction time for a class of nonlinear parabolic equations, Communication Part. *Diff.* Eqn. 4 (1979) 1213-1231.

23. Evans, L. C., Differentiability of a nonlinear semi-group in L^1, J. of Math Analysis and Application 60 (1977) 703-715.

24. Evans, L. C., Application of nonlinear semi-group theory to certain partial differential equations, in: Nonlinear Evolution Equations (ed. M. G. Crandall) Acad. Press (1978).

25. Herrero, M. A. and Vazquez, J. L., Asymptotic behaviour of the solutions of a strong nonlinear parabolic problem, Anales Faculté des Science, Toulouse, Vol III, (1981) 113-127.

26. Ladyzenskaja, O. A. and Solounikov, N. N., Uralčeva, Linear and Quasilinear Equations of Parabolic type, Amer. Math. Cos., Providence, 1968.

27. Lions, P. L., Problémes ellipitiques du 2éme ordre non sous forme diverence.

28. Protter, M. H. and Weinberger, H. F., Maximum Principles in Differential Equations, Prentice Hall, 1967.

29. Sabanina, E. S., A class of nonlinear degenerate parabolic equations, Sov. Math. Doklady 143 (1962) 495-498.

30. Sacks, P., Existence and regularity of the solutions of the inhomogeneous porous medium type equation, TSR #2214, Math Research Center, Madison, Wisconsin.

31. Sacks, P., Continuity of solution of a singular parabolic equation (to appear).

32. Schroeder, Gary (to appear).

33. Vazquez, J. L., Asymptotic behaviour and propagation properties of the one-dimensional flow of gas in porous medium, to appear in Trans. Amer. Math. Soc.

YCK/ed

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 2727	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ASYMPTOTIC BEHAVIOUR OF THE PLASMA EQUATION		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Y. C. Kwong		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office, P.O. Box 12211, Research Triangle Park, North Carolina 27709		12. REPORT DATE August 1984
		13. NUMBER OF PAGES 53
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

nonlinear degenerate diffusion equation, plasma equation, extinction time, stability, nonlinear semi-group, interior and boundary regularity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, we are concerned with the plasma equation $v(x,t)_t = \Delta v(x,t)^{\frac{1}{q-1}}$ where $q > 2$, $t > 0$, $x \in \Omega$, Ω being a bounded smooth domain in R^N, with non-negative initial data and a homogeneous Dirichlet boundary condition. It is known that there exists a finite extinction time T^* such that the solution decays to zero at T^*. Recently,

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ABSTRACT (cont.)

Berryman and Holland investigated the stability of the profile of the solution as $t \to T^*$. However, they obtained their results at the expense of some very strong regularity assumptions. In this paper, we prove the same kind of results without those strong regularity assumptions. By invoking both the nonlinear "semi-group" theory and a standard regularizing scheme for the equation, we measure the rate of decay of the solution and obtain estimates on the time derivative as $t \to T^*$. As motivated by the regularity assumptions, both the interior and boundary regularities of the solution are studied. Finally, we perturb the nonlinearity of the plasma equation and study the same aspects for the perturbed equation.

END

FILMED

12-84

DTIC